

Pair-Density-Wave Superconducting States and Electronic Liquid Crystal Phases

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In conventional superconductors the Cooper pairs have a zero center of mass momentum. In this paper we present a theory of superconducting states where the Cooper pairs have a nonzero center of mass momentum, inhomogeneous superconducting states known as a pair-density-waves (PDW) states. We show that in a system of spin-1/2 fermions in 2 dimensions in an electronic nematic spin triplet phase where rotational symmetry is broken both in real and in spin space PDW phases arise naturally in a theory that can be analyzed using controlled approximations. We show that several superfluid phases that may arise in this phase can be treated within a controlled BCS mean field theory, with the strength of the spin-triplet nematic order parameter playing the role of the small parameter of this theory. We find that in a spin-triplet nematic phase, in addition of a triplet p -wave and spin-singlet d -wave (or s depending on the nematic phase) uniform superconducting states, it is also possible to have a d -wave (or s) PDW superconductor. The PDW phases found here can be either unidirectional, bidirectional or tridirectional depending on the spin-triplet nematic phase and which superconducting channel is dominant. In addition, a triple-helix state is found in a particular channel. We show that these PDW phases are present in the weak coupling limit, in contrast to the usual Fulde-Ferrell-Larkin-Ovchinnikov phases which require strong coupling physics in addition to a large magnetic field (and often both).

I. INTRODUCTION

The problem of the interplay between superconductivity and other broken symmetry states is one of the central problems in the physics of strongly correlated systems. This issue is particularly pressing in the context of the cuprate high temperature superconductors and their complex phase diagram. In addition to Néel antiferromagnetic order and high T_c uniform $d_{x^2-y^2}$ superconductivity, a host of other ordered phases, including incommensurate spin stripes (which exhibit spin-density-wave (SDW) order), incommensurate charge stripes (with charge-density-wave (CDW) order), electronic nematic order, and time-reversal (and/or mirror-plane) symmetry-breaking have been reported essentially in all the cuprate high temperature superconductors.¹⁻⁴ Static spin stripe order is seen in the lanthanum family of the cuprate superconductors.⁵⁻⁷ Static charge stripe order is seen in $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$ (LBCO),⁸ in $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$ (YBCO),^{9,10} in high magnetic fields (where otherwise is seen as short range order¹¹⁻¹³), and in $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ (BSCCO).¹⁴⁻¹⁶ Nematic charge order is seen in YBCO¹⁷⁻¹⁹ and in BSCCO²⁰ over a wide range of doping and temperatures. Time-reversal and/or mirror plane (or inversion) symmetry breaking has also been reported in YBCO, in LBCO and in BSCCO²¹⁻²⁴ although recent NMR measurements do not detect magnetism in the same samples.²⁵ Stripe and/or nematic orders of these types are also seen in the iron superconductors²⁶⁻²⁸ and in heavy fermion materials.^{29,30}

A key feature of the orders that are seen in these strongly correlated materials is that the orders are intertwined with each other rather instead of competing with each other.^{31,32} By intertwined orders what we mean⁴

is that the orders appear either together and/or with similar strengths, e.g. at critical temperatures of similar magnitude, over a significant range of parameters (doping, coupling constants, etc.) Instead, if the orders were competing with each other, one of the orders will be stronger and the others will be strongly suppressed. The exception to this rule are systems which are close to a multicritical point at which not only the critical temperatures but also all the couplings between the different orders are finely-tuned to very specific relations (and values). While this can happen in a particular material at a particular doping it is unnatural to assume that multicriticality should generically occur in all materials and for a wide range of parameters.

A case that is particularly relevant from the perspective of intertwined orders is LBCO, particularly near the so-called 1/8 anomaly. In this material the T_c of the uniform d -wave superconductivity is suppressed (down to low-temperatures). Yet, a variety of experimental probes show that over essentially the same temperature range where at other dopings LBCO is a d -wave superconductors, near 1/8 doping a host of other orders are observed, including charge-stripe order, spin stripe order and a most peculiar phase in which the CuO planes appear to be superconducting but yet the material remains insulating along the c -axis.^{33,34} The layer-decoupling effect is also seen in LBCO away from $x = 1/8$ at finite fields³⁵ and also in underdoped LSCO materials at finite magnetic fields^{36,37} where a field-induced stripe-ordered state had been observed previously.⁶

It was suggested by Berg and coworkers that this peculiar layer-decoupling effect can be naturally explained if the CuO planes are in an inhomogeneous, striped, superconducting state with the symmetry of a pair-density wave (PDW) state in which charge, spin and supercon-

ducting orders are intertwined with each other.^{31,38} The local superconducting order parameter $\Delta(\mathbf{r})$ in a PDW state is spatially modulated and a spin singlet. For a state with unidirectional modulation, $\Delta(\mathbf{r})$ has the form

$$\Delta(\mathbf{r}) = \Delta_{\mathbf{Q}}(\mathbf{r}) e^{i\mathbf{Q}\cdot\mathbf{r}} + \Delta_{-\mathbf{Q}}(\mathbf{r}) e^{-i\mathbf{Q}\cdot\mathbf{r}} \quad (1.1)$$

where $\Delta_{\pm\mathbf{Q}}(\mathbf{r})$ are two slowly-varying complex fields and \mathbf{Q} is the ordering wave vector. Hence, the unidirectional PDW superconducting state is characterized by two complex order parameters, $\Delta_{\pm\mathbf{Q}}(\mathbf{r})$.

A state with the PDW pattern of superconducting order was proposed already in 1964 by Larkin and Ovchinnikov³⁹ (LO) and by Fulde and Ferrell⁴⁰ (FF) to arise in the presence of a Zeeman field. As is the case of all ordered phases with a finite wave vector, the LO state, and its time-reversal breaking (spiral) cousin FF, requires that a nesting condition be satisfied for this state to occur in the weak coupling BCS regime (for a review on FFLO states see Ref. [41]). In most cases this nesting conditions is hardly ever satisfied. Thus, states of this type can only exist in a strongly coupled regime which is clearly outside the applicability of a weak coupling theory such as BCS.

Motivated by the LBCO results, and using the BCS framework, Loder and coworkers⁴² found a PDW state in a tight-binding model with d -wave pairing in the absence of an external magnetic field. However, these authors found that the critical value of the coupling constant for which the PDW is the ground state is quite large and hence well outside the regime in which BCS theory is reliable. More recently, a PDW state has been found in variational Monte Carlo simulations of the $t - J$ and $t - t' - J$ model at zero magnetic field,⁴³⁻⁴⁶ although in these simulations appear to favor the uniform SC state over the PDW state only by a small amount of energy. However, recent, sophisticated iPEPS (infinite projected entangled pair-states⁴⁷) simulations have found strong evidence for intertwined orders (in which several orders, including the PDW state, appear to be essentially degenerate in energy) in the $t - J$ model over a significant range of coupling constants and doping.^{48,49} A recent paper by P. A. Lee (which appeared as this work was being finished) suggests that PDW states may arise in a slave-particle RVB approach⁵⁰ by postulating an “Amperian” interaction among the spinons.⁵¹ On the other hand, a PDW state is known to exist in the spin-gap state of the Kondo-Heisenberg chain⁵² and also in a two-leg ladder,⁵³ even in the weak coupling limit.

FFLO states have been proposed to explain some of the properties of heavy fermion superconductors⁵⁴ and have been conjectured to arise in cold atomic systems.⁵⁵ FFLO states were studied in two dimensions by Shimahara⁵⁶⁻⁵⁸ where the FFLO states seem to be more robust.

Here we will investigate the relation between PDW states and nematic order. Although charge nematic order (a spatially-uniform spin-singlet state that breaks rotational invariance) does neither favor nor disfavor superconductivity, except in regimes in which s -wave and

d -wave superconductivity are in close competition⁵⁹⁻⁶¹ (see, however, Ref.[62]), here we will show that a nematic state in the spin triplet channel⁶³ can favor unconventional superconducting phases, including a PDW state. In this work we present the study of the presence of an inhomogeneous superconducting instability in a system that is already in an α or β nematic phase. We will use a mean field analysis in the weak coupling limit to show that in a region of the phase diagram, an inhomogeneous superconducting state is the ground state of the system.

Oganesyan and coworkers⁶⁴ (as well as Refs.[65 and 66]) studied a spinless Nematic Fermi fluid (FL), where the breaking of rotational symmetry manifest in a spontaneous quadrupolar (elliptical) distortion of the Fermi surface, while the translation invariance is preserved (for a review see Ref.[4]). In the charge nematic state the FS has a spontaneous quadrupolar (elliptical) distortion. Nematic phases of Fermi fluids can arise either via a Pomeranchuk instability of a Fermi liquid^{4,64} or by quantum melting of charge stripe phases.⁶⁷ The resulting anisotropic fluids are non-Fermi liquids if the lattice effects are weak enough.

Wu et al.⁶³ generalized the aforementioned work of Oganesyan and coworkers to a system of spin-1/2 fermions and found a generalization of the nematic state to the spin triplet channel which they called an α -phase. In this phase rotational symmetry is broken both in real and in the internal spin space, while while remaining invariant under a combination of a discrete set of rotations in both sectors. In addition, they also found another, spatially isotropic phase, which they called the β -phase (in analogy to the B phase in liquid ^3He). This state is uniform and spatially isotropic, but the spin quantization axis of a fermionic quasiparticle on the Fermi surface lies in-plane and winds around the FS with an integer-valued winding number. In both phases the FS for spin up and down is distorted in different ways (see Figs. 1(a), 1(b), and 1(c)) providing a natural system to studied the presence of an instability to an inhomogeneous superconducting state. In a Fermi liquid setting, the phase transition to the spin triplet nematic phases occurs as a Pomeranchuk instability and hence the tuning parameter is a Landau parameter in the spin triplet channel. In a strong coupling setting it can occur by quantum melting of a spin-stripe state. In what follows we will refer to both the α and the β phases as spin-triplet nematic phases (although in a strict sense they are not).

In the conventional BCS approach^{39,40} the FFLO states arise only in a regime in which there is a sufficiently weak Zeeman coupling to an uniform magnetic field so that the SC instability can only occur for Cooper pairs with finite momentum by suppressing the nesting between electronic states at the Fermi surfaces for both spin projections. However, this assumption is a severe limitation and, to this date, Zeeman-field-tuned FFLO states have not been clearly seen in experiment. In contrast here we will see that in the spin triplet nematic phases (which although magnetic have a zero uniform

Zeeman field) the tuning parameter for the SC instability is the distance to the nematic spin triplet quantum critical point. In particular we will find that depending on whether the nematic is an α or a β phase a host of different SC states, both uniform and inhomogeneous, can occur.

Unfortunately to this date there is no clear evidence for a spin triplet nematic state. On the theoretical side a recent paper by Maharaj and coworkers⁶⁸ found a spin-triplet β -phase in a fermionic system on a honeycomb lattice via a Pomeranchuk instability. Fischer and Kim found a nematic-spin-nematic state (the α spin-triplet nematic state) in a mean-field analysis of the three-band Emery model of the cuprates in a regime in which the Hubbard U_d on the Cu sites and on the O sites (U_p) are both large (and comparable).⁶⁹ On the experimental side, there is evidence of time reversal-symmetry-breaking in $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$ close to the pseudogap temperature in spin-polarized neutron scattering^{21,24} and, with some caveats, in Kerr rotation experiments.^{22,23} However, the Kerr rotation experiments can also be interpreted as evidence of inversion symmetry breaking via a gyrotropic effect in a system with charge order.⁷⁰ Hence the Kerr effect measurements do not on their own prove the existence of a state with broken time reversal invariance since the cuprate superconductors are now known to exhibit charge order. On the other hand, the spin-polarized neutron experiments can be interpreted either as evidence for loop current order⁷¹ or as evidence of a nematic spin triplet state which on a CuO lattice means that the oxygens are spin-polarized but their polarization is opposite along the a and b axis (as shown in Fig. 1(d)). However such a state is incompatible with NMR measurements which do not find evidence of any sublattice magnetization in YBCO and HBCO which have instead a substantial spin gap.

Aside from these important caveats and reservations, we find that it is nevertheless useful to consider the possible role of spin triplet nematic phases in a weak-coupling mechanism for pair-density-wave phases. In this work we will consider a system in a spin triplet nematic state but close to the Pomeranchuk quantum critical point. By restricting ourselves to this regime enable us to use controlled approximations. We will assume that the system of interest is inside a spin-triplet nematic state, sufficiently close to the quantum phase transition so that the magnitude of the order parameter. However we will also assume that we are deep enough in the spin-triplet nematic phase so that the quantum critical fluctuations can be safely ignored. Furthermore we will also ignore the possible non-Fermi liquid physics which may arise in the spin-triplet nematic state. Thus, the main assumption that we will use throughout is the existence to the Pomeranchuk quantum critical point and that the resulting α and β phases are stable. For this reason we will not consider the $l = 1$ case since these phases are unstable in the absence of sufficiently strong spin-orbit interactions.⁶³ We will show that, depending on

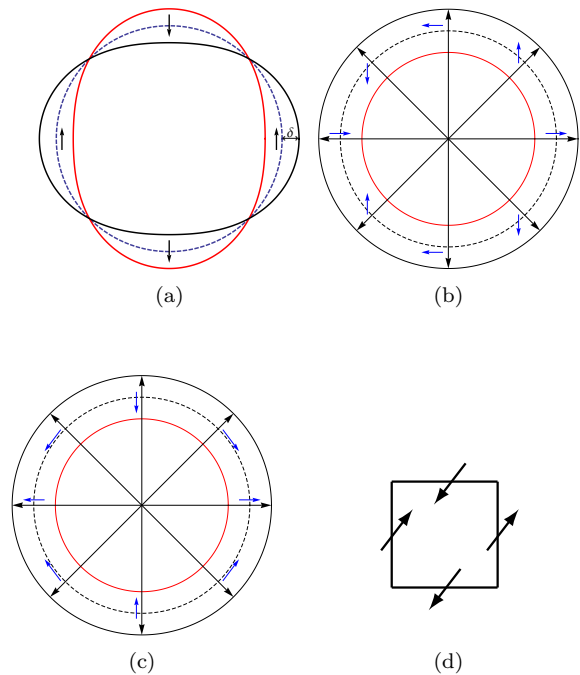


FIG. 1. (a) Distortion of the FS for the fermions with spin up (black) and down (red) for a triplet α nematic phase with $l = 2$ (Wu et al.⁶³). (b),(c) Distortion of the FS for the fermions with spin plus (black) and minus (red) with respect to the spin quantization axis (blue) for a β nematic phase with $l = 2$ and $l = -1$ respectively. (d) Putative nematic spin order in a copper oxide plane in YBCO.²¹

the particular spin triplet nematic phase that is considered, different uniform superconducting phases arise (s , p or d wave) and that these phases are in close competition with inhomogeneous phases with the symmetry of a pair-density-wave of the LO type. FF states are generally found to be metastable at least close to the thermal phase boundary.

The main results of this work are summarized in three phase diagrams, one for the spin triplet nematic α phase with pairing in the d -wave superconducting channel (shown in Fig.4(a)) and two for the spin triplet nematic β phase with pairing in the s and d wave superconducting channels (shown in Fig.5 and Fig.6 respectively.) We also determine the structure of the Landau-Ginzburg free energies close to the thermal transition and calculate the coefficients and stiffnesses. The resulting phase diagrams turn out to be quite complex. In the case of the α phase the superconducting states which arise are, in addition to a spin-triplet p wave state, a uniform spin singlet d -wave SC, a bidirectional PDW state, and a unidirectional PDW state. On the other hand, in the case of the β phase the uniform state may be an s -wave or a d wave SC. If the pairing channel is s wave, in the β phase we find unidirectional, bidirectional and tridirectional PDW states and, in addition, a triple-helix FF-type state. If the pairing channel is d -wave, in addition to an uniform d -wave SC, we also find both a unidirec-

tional and two bidirectional PDW phases. We also investigate the nature of the phase transitions between these states close to the thermal phase boundary. A rich set of different behaviors are found, including continuous and first order phase transitions as well as Lifshitz points and other multicritical points. It is important to emphasize that these results, obtained using a weak coupling BCS theory, are controlled by the distance to the spin triplet nematic quantum critical point. Thus the spin triplet nematic quantum critical point plays the role of a complex multicritical point.

This paper is organized as follows. In Section II we summarize the theory and description of the spin-triplet nematic phases and follow closely the results and notation of Ref. [63]. This caveats are discussed in this section in some detail. In Section III we discuss the SC instabilities of the α (Subsection III A) and β (Subsection III B) phases by calculating explicitly the respective SC susceptibilities. In Section IV we present a BCS-type mean-field theory of the different SC states and show that it is well controlled in the regime where the spin-triplet nematic order parameter is small enough. In this Section we derive the Landau-Ginzburg free energy for each phase and derive the phase diagrams and in Section V we present our conclusions. The details of the calculations are presented in the Appendix.

II. SPIN-TRIPLET NEMATIC PHASES

We start by recalling some of the main results on spin-triplet nematic phases in two dimensions from Ref. [63] which are relevant for the present work. The mean-field (MF) Hamiltonian⁶³ for a spin-triplet nematic phase is:

$$H = \sum_{\mathbf{k}} c_{\mathbf{k},\alpha}^\dagger \{ \epsilon_{\mathbf{k}} - [\mathbf{n}_1 \cos(l\theta) + \mathbf{n}_2 \sin(l\theta)] \cdot \boldsymbol{\sigma}_{\alpha,\beta} \} c_{\mathbf{k},\beta} + \frac{|\mathbf{n}_1|^2 + |\mathbf{n}_2|^2}{2|f_l^a|} \quad (2.1)$$

where \mathbf{n}_1 and \mathbf{n}_2 are the order parameters for the spin-triplet nematic phase, $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ are the three 2×2 Pauli matrices, $l \in \mathbb{Z}$, θ is the polar angle between \mathbf{k} and the k_x axis and f_l^a are the Landau parameters in the spin triplet channel of Fermi liquid theory.⁷²

The order parameter fields \mathbf{n}_1 and \mathbf{n}_2 transform under a global $SO(3)_S$ rotation R in the spin channel (denoted here by S) as follows

$$\mathbf{n}_1 \mapsto R \cdot \mathbf{n}_1, \quad \mathbf{n}_2 \mapsto R \cdot \mathbf{n}_2 \quad (2.2)$$

In addition, the order parameter fields \mathbf{n}_1 and \mathbf{n}_2 transform as follows under a spatial rotation by a global angle θ about the z axis perpendicular to the 2D plane

$$\begin{aligned} \mathbf{n}_1 &\mapsto \cos(l\theta)\mathbf{n}_1 + \sin(l\theta)\mathbf{n}_2 \\ \mathbf{n}_2 &\mapsto -\sin(l\theta)\mathbf{n}_1 + \cos(l\theta)\mathbf{n}_2 \end{aligned} \quad (2.3)$$

We will refer to this as the $SO(2)_L$ “orbital” (or spatial) rotational invariance. This symmetry is exact in

an electron fluid in the continuum and reduces to a discrete subgroup for a lattice model, i.e. the point or space group of the lattice, and it is contained in the symmetries of the free-fermion band structure denoted in Eq.(2.1) by $\epsilon_{\mathbf{k}}$. For simplicity in this paper we will consider an electron fluid in the continuum in which case $\epsilon_{\mathbf{k}}$ is invariant under $SO(2)_L$ rotations.

The Ginzburg-Landau (GL) free energy for the system must be invariant under the global combined symmetry $SO(2)_L \otimes SO(3)_S$. We will focus first in the dependence of the GL free energy for phases in which the order parameter fields \mathbf{n}_1 and \mathbf{n}_2 take uniform values, and hence do not depend on the position \mathbf{x} . Under this assumption, to low orders in the order parameter fields, the most general $SO(2)_L \otimes SO(3)_S$ -invariant form of the GL free energy is given by:

$$\begin{aligned} F(\mathbf{n}_1, \mathbf{n}_2) = & r(|\mathbf{n}_1|^2 + |\mathbf{n}_2|^2) + v_1(|\mathbf{n}_1|^2 + |\mathbf{n}_2|^2)^2 + v_2|\mathbf{n}_1 \times \mathbf{n}_2|^2 \\ & + \dots \end{aligned} \quad (2.4)$$

where r , v_1 and v_2 are three parameters (or coupling constants). As usual r is a linear measure of the distance to the critical temperature (for the thermal transition) or to the critical coupling constants (e.g. the Landau parameters f_l) in the case of the quantum phase transition.

For $r < 0$ the system is in a broken symmetry state, and the GL free energy in Eq. (2.4) has two type of solutions depending on the sign of v_2 . For $v_2 > 0$ it is most favorable to have a state where $\mathbf{n}_1 \parallel \mathbf{n}_2$. This is the α -phase.⁶³ On the other hand, for $v_2 < 0$ it is most favorable to have a state where $\mathbf{n}_1 \perp \mathbf{n}_2$ and $|\mathbf{n}_1| = |\mathbf{n}_2|$. This is the β -phase.⁶³

In the α -phase the Fermi surface (FS) of the electrons with spin up and down become spontaneously anisotropic in space. Hence in this phase both $SO(2)_L$ and $SO(3)_S$ are spontaneously broken symmetries. In this phase, we can choose $\mathbf{n}_1 = \bar{n}\hat{z}$ and $\mathbf{n}_2 = 0$ (notice that we can get a non zero \mathbf{n}_2 just doing a rotation around the z axis, so this is always allowed). However, in the α phase the system retains the discrete unbroken symmetry of spatial rotations by π/l combined with a global spin flip. On the other hand, the β -phase corresponds to a phase where the spin polarization axis winds around the FS. Here we choose $|\mathbf{n}_1| = |\mathbf{n}_2| = \bar{n}$ and $\mathbf{n}_1 = \bar{n}\hat{x}$ and $\mathbf{n}_2 = \bar{n}\hat{y}$ (which can always be achieved by a rotation in spin space).

In the following sections we will discuss the SC instabilities (and phases) which arise in these α and β phases. To this end, in addition to the Hamiltonian in Eq. (2.1), we will add a pairing interaction in the spin-singlet channel of the form⁷³

$$H_p = \sum_{\mathbf{k}, \mathbf{k}'} V(\mathbf{k}, \mathbf{k}') c_{\mathbf{k}+\mathbf{q}/2, \uparrow}^\dagger c_{-\mathbf{k}+\mathbf{q}/2, \downarrow}^\dagger c_{-\mathbf{k}'+\mathbf{q}/2, \downarrow} c_{\mathbf{k}'+\mathbf{q}/2, \uparrow} \quad (2.5)$$

where

$$V(\mathbf{k}, \mathbf{k}') = -g_\lambda \gamma_\lambda(\hat{\mathbf{k}}) \gamma_\lambda(\hat{\mathbf{k}}') \quad (2.6)$$

where g_λ is the coupling constant in the channel labeled by λ , and $\gamma_\lambda(\hat{\mathbf{k}})$ is the normalized form factor of the λ channel (e.g. λ can correspond to s, d, \dots wave pairing) and obey the normalization condition

$$\int \frac{d\theta}{2\pi} \gamma_\lambda^2(\hat{\mathbf{k}}) = 1 \quad (2.7)$$

For instance, the s -wave and d -wave form factors are

$$\begin{aligned} \gamma_s(\hat{\mathbf{k}}) &= 1, & (s - \text{wave}) \\ \gamma_{d_{x^2-y^2}}(\hat{\mathbf{k}}) &= \sqrt{2}(\hat{k}_x^2 - \hat{k}_y^2) = \sqrt{2} \cos 2\theta & (d - \text{wave}) \end{aligned} \quad (2.8)$$

As usual, the s -wave form factor is nodeless while the d -wave form factor has nodes at $\theta = (2n+1)\pi/4$, where $n \in \mathbb{Z}$.

We will show below that there are SC instabilities at critical values of the coupling constants g_λ^c , which are controlled (tuned) by the expectation value of the spin-triplet nematic order parameter, denoted above by \bar{n}

which, in turn, is determined by how far the system is into a spin triplet nematic state from its quantum critical point to the normal Fermi fluid. In particular we will see that for \bar{n} small enough there are SC instabilities in the weak coupling regimes of these coupling constants. Therefore, the theory we are presenting in this work can be regarded as a theory of a multicritical point for a system close to spin-triplet nematic phases and superconducting phases (both uniform and non-uniform).

III. SUPERCONDUCTING INSTABILITIES

We start by looking at the Cooper instability in the s -wave and d -wave channels for both the α - and β -phases in each of the spin triplet nematic phases. We begin by writing down the SC susceptibility (i.e. the bubble diagram in the particle-particle channel) of the isotropic electron fluid $\chi_{sc}(\mathbf{Q}, i\omega_m)$,

$$\chi_{sc}(\mathbf{Q}, i\omega_m) = T \sum_{n=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2} \gamma_\lambda^2(\hat{\mathbf{k}}) G_0(\mathbf{k} + \mathbf{Q}/2, i\omega_n + i\omega_m/2) G_0(-\mathbf{k} + \mathbf{Q}/2, -i\omega_n + i\omega_m/2), \quad (3.1)$$

where $\omega_n = (2n+1)\pi T$ are fermionic Matsubara frequencies, $\omega_m = 2m\pi T$ are bosonic Matsubara frequencies, and

$$G_0(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon(\mathbf{k})} \quad (3.2)$$

is the free-fermion Green function. After performing the Matsubara sum in Eq. (3.1) we obtain

$$\begin{aligned} \chi_{sc}(\mathbf{Q}, i\omega_m) &= \\ &= \int \frac{d^2k}{(2\pi)^2} \gamma_\lambda^2(\hat{\mathbf{k}}) \frac{1 - n_F(\epsilon(\mathbf{k} + \mathbf{Q}/2)) - n_F(\epsilon(-\mathbf{k} + \mathbf{Q}/2))}{\epsilon(\mathbf{k} + \mathbf{Q}/2) + \epsilon(-\mathbf{k} + \mathbf{Q}/2) - i\omega_m} \end{aligned} \quad (3.3)$$

and

$$n_F(\epsilon) = \frac{1}{e^{\epsilon/T} + 1} \quad (3.4)$$

is the Fermi-Dirac distribution.

At finite temperature, Eq. (3.3) in general has to be evaluated numerically. However, at zero temperature it is possible to obtain explicit analytic expressions for the SC susceptibility. Below, we will focus first on the zero temperature SC instabilities and we will take $\omega_m = 0$. In this case we find

$$\begin{aligned} \chi_{sc}(\mathbf{Q}) &= \\ &= \int \frac{d^2k}{(2\pi)^2} \gamma_\lambda^2(\hat{\mathbf{k}}) \frac{1 - \Theta(-\epsilon(\mathbf{k} + \mathbf{Q}/2)) - \Theta(-\epsilon(-\mathbf{k} + \mathbf{Q}/2))}{\epsilon(\mathbf{k} + \mathbf{Q}/2) + \epsilon(-\mathbf{k} + \mathbf{Q}/2)} \end{aligned} \quad (3.5)$$

We will evaluate Eq. (3.5) for both the α - and β -phases.

A. α -phase

From now on we will focus in the (quadrupolar) $l = 2$ channel. In this state, the system remains invariant under a spatial rotation of $\pi/2$ followed by a global spin flip. The α phase is represented by the choice $\mathbf{n}_1 = \delta \hat{\mathbf{z}}$ and $\mathbf{n}_2 = 0$. Hereafter we will use the notation $\bar{n} \rightarrow \delta$, to explicitly state that in the α phase the Fermi surfaces of the up and down spin fermions are distorted as shown in Fig. 1(a)), with δ being the distortion. Notice that from Eq. (2.4), for the α -phase ($v_2 > 0$) we have that:

$$F = r\delta^2 + v_1\delta^4 + \dots \quad (3.6)$$

which has a minimum at $\delta = \sqrt{|r|/2v_1}$. We can see that δ scales with the distance to the quantum critical point. Therefore we can control δ , controlling the parameter r . Keeping that in mind we can write the superconducting susceptibility at wave vector \mathbf{Q} in the α phase in the SC channel λ , $\chi_\alpha^\lambda(\mathbf{Q})$, in the form

$$\frac{\chi_\alpha^\lambda(\mathbf{Q})}{N(E_F)} = \int_0^{2\pi} \frac{d\theta}{2\pi} \gamma_\lambda^2(\hat{\mathbf{k}}) \ln \left| \frac{\omega_D}{\delta \cos(2\theta) - \frac{Q}{2} \cos(\theta - \phi)} \right| \quad (3.7)$$

where $\gamma_\lambda(\hat{\mathbf{k}})$ are the form factors for the s and d wave pairing channels defined in Eq.(2.8). To get the previous

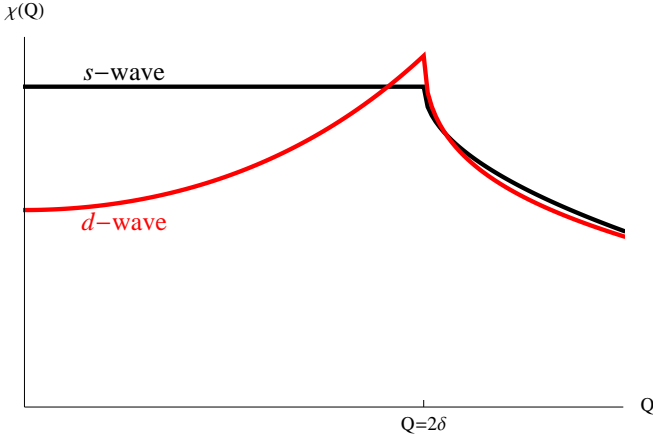


FIG. 2. SC susceptibility for the α -phase in the s -wave and d -wave channels in the direction $\phi = n\pi/2$.

expression we have used the notation

$$\int \frac{d^2k}{(2\pi)^2} \rightarrow N(E_F) \int_{-\omega_D}^{\omega_D} d\xi \int_0^{2\pi} \frac{d\theta}{2\pi} \quad (3.8)$$

and then integrated over the excitation energy ξ (measured from the undistorted FS), where ω_D is an energy cutoff. Here Q and ϕ are the magnitude and the polar angle of the momentum \mathbf{Q} (with \mathbf{Q} being the center of mass momentum of the Cooper pairs) and $N(E_F)$ the density of states on the FS, which will be assumed to be constant.

The susceptibility for the s -wave and d -wave channels in the direction $\phi = n\pi/2$ are plotted in Fig. 2 as a function of $Q = |\mathbf{Q}|$. It is important to note here that while in the s -wave channel there is no preference for a finite value of Q , there is a clear preference in the d -wave channel, where the susceptibility is enhanced at $Q = 2\delta$ and $\phi = n\pi/2$. This provides a first evidence that at least at zero temperature an inhomogeneous superconducting state can be the ground state of the system. From now on we will focus on the d -wave channel, since we are interested in an inhomogeneous superconducting state.

The SC susceptibility for s and d channels at $\mathbf{Q} = 0$ are finite in the α spin triplet nematic state and are given by

$$\begin{aligned} \chi_\alpha^s(0) &= N(E_F) \ln \left(\frac{2\omega_D}{\delta} \right) \\ \chi_\alpha^d(0) &= N(E_F) \ln \left(\frac{2\omega_D}{e^{1/2}\delta} \right) \end{aligned} \quad (3.9)$$

In contrast with the case of a Fermi liquid in which the SC susceptibilities are divergent at $\mathbf{Q} = 0$ (due to the nesting property of the Fermi surface), in the spin triplet nematic α phase they are finite. Consequently in the α phase there is a (mean field theory) critical value of the pairing coupling constants g_s and g_d for the uniform SC state to occur,

$$g_s^c = \chi_\alpha^s(0)^{-1}, \quad g_d^c = \chi_\alpha^d(0)^{-1} \quad (3.10)$$

The value of the susceptibility at $Q = 2\delta$ can be determined evaluating Eq. (3.7),

$$\chi_\alpha^d(\mathbf{Q}_{op}) = N(E_F) \ln \left(\frac{2\omega_D e^{1/8}}{\delta} \right) \quad (3.11)$$

where $|\mathbf{Q}_{op}| = 2\delta$ and \mathbf{Q}_{op} points in the $n\pi/2$ direction. The (mean field theory) critical value of the coupling constant in order to have a Cooper instability at finite \mathbf{Q} in the d wave channel is

$$g_{dc}^\alpha(\mathbf{Q}_{op}) = \chi_\alpha^d(\mathbf{Q}_{op})^{-1} \quad (3.12)$$

In the d -wave case there is an extra factor of $e^{1/8}$ that is not present in the s -wave channel. This extra factor reduces the critical value of the coupling constant in the d -wave channel.

An important feature of the result of Eq.(3.12) is that the value of g_{dc} is controlled by the magnitude $\bar{n} = \delta$ of the spin-triplet nematic state which, more geometrically, parametrizes the distortions δ of the Fermi surfaces for fermions with up and down spins. It is the smallness of the parameter δ that allows us to work in the weak coupling regime and hence to use BCS theory when δ is very small. This result will be extended in the next Section to finite temperature where it will be used to determine the phase diagram.

Finally let us discuss briefly the role of spin-triplet pairing interactions (e.g. p -wave pairing). In contrast to what we found in the singlet s and d wave channels, the Fermi surfaces of the α phase are still nested. As a result, there is an infinitesimal SC instability in the uniform p -wave channel. However, provided we assume that the coupling constant for this pairing channel is sufficiently weak, the T_c for the d -wave channel is always higher than the T_c for the p -wave channel. In what follows we will ignore the p -wave channel.

In conclusion, in the α phase there is a critical value of the pairing coupling constant for both the s - and d -wave uniform SC channels. However, the s -wave channel does not favor the formation of SC states with finite wave vector whereas the d -wave channel clearly does, as shown in Fig.2. In what follows we will only consider the case of the d -wave channel.

B. β -Phase

From Eq. (2.4) for the β -phase ($v_2 < 0$) we have that:

$$F = 2r\bar{n}^2 + 4v_1\bar{n}^4 + v_2\bar{n}^4 + \dots \quad (3.13)$$

which has a minimum at $\bar{n} = \sqrt{|r|/(4v_1 + v_2)}$. We can see that \bar{n} scales with the distance to the quantum critical point. Therefore, we can control \bar{n} , controlling the parameter r . As for the α -phase we start by looking at the Cooper instability in the β -phase. Since in the β -phase this case the FS's are spherically symmetric (see Figs. 1(b) and 1(c)), the SC susceptibility in the pairing channel λ at finite temperature T (Eq. (3.3) with $\omega_m = 0$) can be written for general l as:

$$\begin{aligned} \frac{\chi_\beta^\lambda(\mathbf{Q}, T)}{N(E_F)} = & \int_{-\omega_D}^{\omega_D} d\xi \int_0^{2\pi} \frac{d\theta}{2\pi} \gamma_\lambda^2(\hat{\mathbf{k}}) \frac{1}{8\xi(\bar{n}-\xi)(\bar{n}+\xi)} \\ & \left[(\bar{n}(-1)^l + \bar{n} - 2\xi)(\bar{n} + \xi) \left(\tanh\left(\frac{\bar{n} - \xi - Q/2 \cos(\theta - \phi)}{2T}\right) + \tanh\left(\frac{\bar{n} - \xi + Q/2 \cos(\theta - \phi)}{2T}\right) \right) \right. \\ & \left. - (\bar{n}(-1)^l + \bar{n} + 2\xi)(\bar{n} - \xi) \left(\tanh\left(\frac{\bar{n} + \xi - Q/2 \cos(\theta - \phi)}{2T}\right) + \tanh\left(\frac{\bar{n} + \xi + Q/2 \cos(\theta - \phi)}{2T}\right) \right) \right] \end{aligned} \quad (3.14)$$

Notice that the expression for the the SC susceptibility in Eq. (3.14) depends only on the parity of l , and not on it's value.

the zero temperature limit.

1. l odd

Let us analyze briefly the behavior of the SC susceptibilities for the l odd and l even cases before discussing

For l odd the expression of the SC susceptibility in pairing channel λ of Eq. (3.14) reduces to:

$$\begin{aligned} \frac{\chi_\beta^\lambda(\mathbf{Q}, T)}{N(E_F)} = & \frac{1}{4} \int_{-\omega_D}^{\omega_D} d\xi \int_0^{2\pi} \frac{d\theta}{2\pi} \gamma_\lambda^2(\hat{\mathbf{k}}) \left[\frac{1}{(\bar{n} - \xi)} \left(\tanh\left(\frac{\bar{n} - \xi - Q/2 \cos(\theta - \phi)}{2T}\right) + \tanh\left(\frac{\bar{n} - \xi + Q/2 \cos(\theta - \phi)}{2T}\right) \right) \right. \\ & \left. + \frac{1}{(\bar{n} + \xi)} \left(\tanh\left(\frac{\bar{n} + \xi - Q/2 \cos(\theta - \phi)}{2T}\right) + \tanh\left(\frac{\bar{n} + \xi + Q/2 \cos(\theta - \phi)}{2T}\right) \right) \right] \end{aligned} \quad (3.15)$$

At $Q = 0$ the previous expression reduces to the BCS result

$$\frac{\chi_\beta^\lambda(0, T)}{N(E_F)} = \int_{-\omega_D}^{\omega_D} d\xi \frac{1}{\xi} \tanh\left(\frac{\xi}{2T}\right) \quad (3.16)$$

where we used that $\omega_D \gg \xi$ and we made a change of variables. We can then deduce that for odd l , the uniform SC state is the most favorable state since there is a logarithmic divergence of its susceptibility at $T = 0$. Notice the close similarity, for example, with the case where there is a finite spin-orbit coupling (see Ref.[74] and references therein). In the case of a Rashba spin-orbit interaction (which is similar to the β -phase with

$l = 1$), the uniform SC state is favorable in the absence of magnetic field. However, in the presence of a Zeeman coupling to a magnetic field, it is possible to favor an inhomogeneous superconducting state (we will not study the effect of magnetic fields in the present paper). Those states have been recently studied extensively by Zhang et. al.⁷⁵⁻⁷⁷

2. l even

In this case, the SC susceptibility of Eq. (3.14) reduces to:

$$\begin{aligned} \frac{\chi_\beta^\lambda(\mathbf{Q}, T)}{N(E_F)} = & \int_{-\omega_D}^{\omega_D} d\xi \int_0^{2\pi} \frac{d\theta}{2\pi} \gamma_\lambda^2(\hat{\mathbf{k}}) \frac{1}{4\xi} \left[1 - \bar{n}_F(\xi + \bar{n} - Q/2 \cos(\theta - \phi)) - n_F(\xi - \bar{n} + Q/2 \cos(\theta - \phi)) \right. \\ & \left. + 1 - n_F(\xi + \bar{n} + Q/2 \cos(\theta - \phi)) - n_F(\xi - \bar{n} - Q/2 \cos(\theta - \phi)) \right] \end{aligned} \quad (3.17)$$

Having determined the expression for finite T , we will compute the SC susceptibility at $T = 0$. After integrat-

ing over ξ in Eq. (3.17) and taking the $T \rightarrow 0$ limit, we

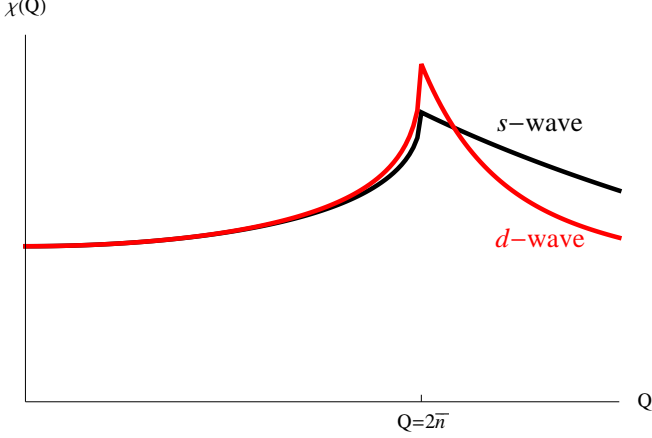


FIG. 3. SC susceptibility for the β -phase in the s -wave and d -wave channels in the direction $\phi = n\pi/2$.

get:

$$\frac{\chi_{\beta}^{\lambda}(\mathbf{Q}, 0)}{N(E_F)} = \int_0^{2\pi} \frac{d\theta}{2\pi} \gamma_{\lambda}^2(\hat{\mathbf{k}}) \ln \left| \frac{\omega_D}{\bar{n} - Q/2 \cos(\theta - \phi)} \right| \quad (3.18)$$

For the s -wave case, $\gamma_s(\hat{\mathbf{k}}) = 1$, and the previous expression can be easily evaluated to be

$$\frac{\chi_{\beta}^s(\mathbf{Q}, 0)}{N(E_F)} = \begin{cases} \ln \left(\frac{2\omega_D/\bar{n}}{1 + \sqrt{1 - (Q/2\bar{n})^2}} \right), & 0 \leq Q \leq 2\bar{n} \\ \ln \left(\frac{4\omega_D}{Q} \right), & Q > 2\bar{n} \end{cases} \quad (3.19)$$

We can see that the value of Q that gives the maximum susceptibility is $Q = 2\bar{n}$ and \mathbf{Q} can point in any direction by rotational symmetry.

On the other hand, for the d -wave case, whose form factor is $\gamma_d(\hat{\mathbf{k}}) = \sqrt{2} \cos 2\theta$, we have to compute numerically the SC susceptibility of Eq.(3.18), and found that the maximum is at $\phi = n\pi/2$ and $Q = 2\bar{n}$, i.e. the antinodal directions of the d -wave order parameter.

Just as in the case of the α -phase, there is a critical value for the pairing coupling constant in the s - and d -wave channels given by the inverse of the respective SC susceptibilities (e.g. Eq.(3.18)). Even though there is a critical value for the coupling constants, this is smaller than the critical value for $Q = 0$. Therefore, for even l , the condensation Cooper pairs with finite momentum is more favorable (at least at low temperatures for both the s - and d -wave channels). Also notice that, as in the α -phase, in the β -phase we also find that the critical pairing coupling constants in the s and d wave channels obey $g_{\beta,c}^d < g_{\beta,c}^s$ since the d -wave channel has a larger SC susceptibility than the s wave channel at the ordering wave vector. Let us mention that basically the same expression for the susceptibility for the s -wave Eq. (3.19) was obtained by Shimahara⁵⁶ who considered an FF phase in

a 2D electron gas in the presence of a Zeeman coupling to a perpendicular magnetic field, h . His expression for the susceptibility differs from us in that our \bar{n} is replaced in his expression by h . At the mean-field level there is a close analogy between the two problems. Here, we can get an inhomogeneous superconducting phase without an external magnetic field, if we have the system in a β -phase with even angular momentum l .

IV. MEAN FIELD THEORY AT $T > 0$

We will now consider the mean-field (MF) theory of a Hamiltonian that includes the nematic phase and the pairing interaction Eq.(2.5). For that, we will work in the imaginary time path integral formalism where the action is given by⁷⁸

$$S = \int_0^{\beta} d\tau \left[\int d\mathbf{x} \bar{\psi}_{\sigma}(\mathbf{x}, \tau) (\partial_{\tau} - \mu) \psi_{\sigma}(\mathbf{x}, \tau) + H(\bar{\psi}, \psi) \right] \quad (4.1)$$

where $\psi_{\sigma}(\mathbf{x}, \tau)$ is a Fermi field for spin-1/2 fermions, μ is the chemical potential, and H is the full Hamiltonian. We will perform a Hubbard-Stratonovich transformation to get rid of the quartic fermionic terms in the pairing term in H . We will consider both the α -phase and the β -phase of the spin-triplet nematic state.

A. α -phase

Let us start by looking at the α -phase. In this case the effective action for the superconducting state is given by:

$$S = \int_0^{\beta} d\tau \left[\sum_{\mathbf{k}, \sigma} \bar{\psi}_{\mathbf{k}, \sigma} (\partial_{\tau} + \xi_{\mathbf{k}, \sigma}) \psi_{\mathbf{k}, \sigma} + \sum_{\mathbf{q}} \frac{|\Delta_{\mathbf{q}}|^2}{g} - \sum_{\mathbf{q}} \sum_{\mathbf{k}} \gamma(\hat{\mathbf{k}}) \bar{\psi}_{\mathbf{k}+\mathbf{q}/2, \uparrow} \bar{\psi}_{-\mathbf{k}+\mathbf{q}/2, \downarrow} \Delta_{\mathbf{q}} - \sum_{\mathbf{q}} \sum_{\mathbf{k}} \gamma(\hat{\mathbf{k}}) \Delta_{\mathbf{q}}^* \psi_{-\mathbf{k}+\mathbf{q}/2, \downarrow} \psi_{\mathbf{k}+\mathbf{q}/2, \uparrow} \right] \quad (4.2)$$

where $\Delta_{\mathbf{q}}(\tau)$ is the Hubbard-Stratonovich field associated with the superconducting order parameter at wave vector \mathbf{q} . In the α -phase the kinetic energies of fermions with up and down spins measured from their respective Fermi surfaces are

$$\xi_{\mathbf{k}, \uparrow} = \xi - \delta \cos 2\theta, \quad \xi_{\mathbf{k}, \downarrow} = \xi + \delta \cos 2\theta \quad (4.3)$$

respectively, where we have included the magnitude of the spin-triplet nematic order parameter δ in the definition of $\xi_{\uparrow, \downarrow}$, and ξ is the energy measured from the undistorted circular FS.

As we saw in the Section III, there are four equivalent directions for which the SC susceptibility for the

α -phase has a maximum, it is natural to focus in the following three different cases for the superconducting order parameters: Fulde-Ferrell (FF), PDW (or Larkin-Ovchinnikov (LO)), bidirectional PDW (or “checkerboard”), and uniform:

- Uniform phases: In the regime in which the α -phase order parameter is very small we find conventional p_x (or p_y) wave (spin-triplet) or $d_{x^2-y^2}$ -wave (spin singlet) (depending on which coupling constant is stronger).
- FF phase: In this phase only one wave vector contributes to the SC order parameter

$$\Delta(\mathbf{r}) = \Delta_{\mathbf{Q}}(\mathbf{r}) e^{i\mathbf{Q} \cdot \mathbf{r}} \quad (4.4)$$

In this phase translation and gauge invariance as well as time reversal and parity are spontaneously broken. The SC order parameter field is a one-component complex field $\Delta_{\mathbf{Q}}(\mathbf{r})$ (which has a constant expectation value).

- PDW phase (or LO phase): two wave vectors contribute to the SC order parameter

$$\Delta(\mathbf{r}) = \Delta_{\mathbf{Q}}(\mathbf{r}) e^{i\mathbf{Q} \cdot \mathbf{r}} + \Delta_{-\mathbf{Q}}(\mathbf{r}) e^{-i\mathbf{Q} \cdot \mathbf{r}} \quad (4.5)$$

This state breaks translation and gauge invariance but it is time-reversal invariant. The order parameter field now has two complex components, $\Delta_{\pm\mathbf{Q}}(\mathbf{r})$ and, hence, has two amplitude fields $|\Delta_{\pm\mathbf{Q}}(\mathbf{r})|$ and two phase fields, $\theta_{\pm\mathbf{Q}}(\mathbf{r}) = \arg[\Delta_{\pm\mathbf{Q}}(\mathbf{r})]$. In the London gauge and with a choice of origin, and with parity invariance $\Delta_{\mathbf{Q}} = \Delta_{-\mathbf{Q}}$, the expectation value of the order parameter takes the LO sinusoidal dependence on position, i.e. $\Delta(\mathbf{r}) = 2|\Delta_{\mathbf{Q}}| \cos(\mathbf{Q} \cdot \mathbf{r})$. The thermal fluctuations of the phase fields $\theta_{\pm\mathbf{Q}}$ play a key role of the thermal melting of the PDW phase.⁷⁹

- Bidirectional phase (or checkerboard): in this phase four wave vectors contribute to the SC order parameter,

$$\Delta(\mathbf{r}) = \Delta_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{r}} + \Delta_{-\mathbf{Q}} e^{-i\mathbf{Q} \cdot \mathbf{r}} + \Delta_{\bar{\mathbf{Q}}} e^{i\bar{\mathbf{Q}} \cdot \mathbf{r}} + \Delta_{-\bar{\mathbf{Q}}} e^{-i\bar{\mathbf{Q}} \cdot \mathbf{r}} \quad (4.6)$$

In this phase the SC order parameter is then a four-component complex field with $\Delta_{\pm\mathbf{Q}}(\mathbf{r})$ and $\Delta_{\pm\bar{\mathbf{Q}}}(\mathbf{r})$ being the four complex components (and hence four amplitudes and four phase fields). Under the assumption of parity and C_4 symmetry it reduces to

$$\Delta(\mathbf{r}) = 2|\Delta_{\mathbf{Q}}| (\cos(\mathbf{Q} \cdot \mathbf{r}) + \cos(\bar{\mathbf{Q}} \cdot \mathbf{r})) \quad (4.7)$$

where $\mathbf{Q} \cdot \bar{\mathbf{Q}} = 0$ and we have assumed $|\Delta_{\mathbf{Q}}| = |\Delta_{-\mathbf{Q}}| = |\Delta_{\bar{\mathbf{Q}}}| = |\Delta_{-\bar{\mathbf{Q}}}|$.

Below we will compute the free energy for each one of these phases.

The free energies of the different states is obtained by integrating out the fermionic degrees of freedom in Eq. (4.2). For the case of FF phase (and for the uniform phases) it is possible to get an explicit expression for the effective free energy as function of the (constant) value of the order parameter field. However, for the PDW and the checkerboard phases this has to be done numerically except near the phase boundary where, if the transition is continuous, the Landau-Ginzburg free energy can be calculated as usual as an expansion in powers of the order parameters.

After writing the fermion operators in the Nambu spinor representation

$$\bar{\Psi}_{\mathbf{k}} = (\bar{\psi}_{\mathbf{k}+\mathbf{Q}/2, \uparrow}, \psi_{-\mathbf{k}+\mathbf{Q}/2, \downarrow}) \quad (4.8)$$

the action for the general state with a static order parameter $\Delta_{\mathbf{Q}}$ has the form

$$S_{\text{eff}}[\Delta_{\mathbf{Q}}] = - \sum_{\mathbf{k}, \mathbf{k}', n} \bar{\Psi}_{\mathbf{k}, n} \mathcal{G}_{\mathbf{k}, \mathbf{k}', n}^{-1} \Psi_{\mathbf{k}', n} + \beta \sum_{\mathbf{Q}} \frac{|\Delta_{\mathbf{Q}}|^2}{g} + \beta \sum_{\mathbf{k}} \xi_{-\mathbf{k}+\mathbf{Q}/2, \downarrow} \quad (4.9)$$

In the case of the FF phase the modes $\Psi_{\mathbf{k}, n}$ with wave vector \mathbf{k} and Matsubara frequency ω_n decouple from each other and as a result the matrix $\mathcal{G}_{\mathbf{k}, \mathbf{k}', n}^{-1}$ is block diagonal. However, this is not the case for the PDW (LO) SC phases in which, due to this mixing, it is not possible to write the free energy in closed form. Nevertheless sufficiently close to the phase boundary with the normal state, the free energy of the α phase for the PDW SC phases can be computed perturbatively in powers of the SC order parameter with each term being represented by a Feynman diagram computed in the normal phase.

1. Free energy of the FF phase

In the FF case, we can write the action in Eq. (4.2) in the simpler form

$$S[\bar{\Psi}, \Psi, \Delta_{\mathbf{Q}}, \Delta_{\mathbf{Q}}^*] = - \sum_{\mathbf{k}, n} \bar{\Psi}_{\mathbf{k}, n} \mathcal{G}_{\mathbf{k}, i\omega_n}^{-1} \Psi_{\mathbf{k}, n} + \beta \frac{|\Delta_{\mathbf{Q}}|^2}{g} + \beta \sum_{\mathbf{k}} \xi_{-\mathbf{k}+\mathbf{Q}/2, \downarrow}, \quad (4.10)$$

where $\beta = 1/T$. Here we assumed that $\Delta_{\mathbf{Q}}$ is constant and real, and we have used the notation

$$\mathcal{G}_{\mathbf{k}, i\omega_n}^{-1} = \begin{pmatrix} i\omega_n - \xi_{\mathbf{k}+\mathbf{Q}/2, \uparrow} & \Delta_{\mathbf{Q}} \gamma(\hat{\mathbf{k}}) \\ \Delta_{\mathbf{Q}}^* \gamma(\hat{\mathbf{k}}) & i\omega_n + \xi_{-\mathbf{k}+\mathbf{Q}/2, \downarrow} \end{pmatrix} \quad (4.11)$$

for the inverse of the fermion Green function in the FF phase, where $\gamma(\hat{\mathbf{k}})$ is the form factor for the different SC channels.

After integrating-out the fermionic degrees of freedom we get

$$S_{\text{eff}}[\Delta_{\mathbf{Q}}, \Delta_{\mathbf{Q}}^*] = -\ln \det[\mathcal{G}^{-1}] + \beta \frac{|\Delta_{\mathbf{Q}}|^2}{g} + \text{const.}, \quad (4.12)$$

We need to compute

$$\ln \det[\mathcal{G}^{-1}] = \sum_{\mathbf{k}, n} \ln(\lambda_{\mathbf{k}, n}^{(1)} \lambda_{\mathbf{k}, n}^{(2)}) \quad (4.13)$$

where $\lambda_{\mathbf{k}, n}^{(i)}$ are the eigenvalues of the matrix $\mathcal{G}_{\mathbf{k}, i\omega_n}^{-1}$. Us-

ing that

$$\begin{aligned} \xi_{\mathbf{k}+\frac{\mathbf{Q}}{2}, \uparrow} &= \xi - \delta \cos 2\theta + \frac{Q}{2} \cos(\theta - \phi) \\ \xi_{-\mathbf{k}+\frac{\mathbf{Q}}{2}, \downarrow} &= \xi + \delta \cos 2\theta - \frac{Q}{2} \cos(\theta - \phi) \end{aligned} \quad (4.14)$$

we find that:

$$\lambda_{\mathbf{k}, n}^{(1,2)} = \pm E_{\mathbf{k}} + \frac{Q}{2} \cos(\theta - \phi) - \delta \cos 2\theta - i\omega_n \quad (4.15)$$

where $E_{\mathbf{k}} = \sqrt{\xi^2 + \gamma^2(\hat{\mathbf{k}})|\Delta_{\mathbf{Q}}|^2}$

After integrating-out the fermionic fields we find

$$\begin{aligned} F_s - F_n &= \frac{|\Delta_{\mathbf{Q}}|^2}{g} - 2TN(E_F) \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{\omega_D} d\xi \left[\ln \left(\frac{1 + e^{-(\sqrt{\xi^2 + 2\cos^2 2\theta} |\Delta_{\mathbf{Q}}|^2 + Q/2 \cos(\theta - \phi) - \delta \cos 2\theta)/T}}{1 + e^{-(\xi + Q/2 \cos(\theta - \phi) - \delta \cos 2\theta)/T}} \right) \right. \\ &\quad \left. + \ln \left(\frac{1 + e^{(\sqrt{\xi^2 + 2\cos^2 2\theta} |\Delta_{\mathbf{Q}}|^2 - Q/2 \cos(\theta - \phi) + \delta \cos 2\theta)/T}}{1 + e^{(\xi - Q/2 \cos(\theta - \phi) + \delta \cos 2\theta)/T}} \right) \right] \end{aligned} \quad (4.16)$$

We can now look for the minimum of the Free energy F_s of Eq.(4.16) with respect to $\Delta_{\mathbf{Q}}$ and \mathbf{Q} to find the thermodynamically stable state. We do this minimization numerically over a range of values for T and δ . For the d -wave channel we find a range of T and δ in which there is superconducting order, $\Delta \neq 0$, but which may also be inhomogeneous and hence has $\mathbf{Q} \neq 0$, as expected from the SC instabilities computed in section III. This result suggests the possible presence of either a time-reversal breaking FF SC state or a time-reversal invariant PDW (or LO) SC state. In addition, the transition from the normal (non-SC state) to the putative FF state is continuous. Since a continuous transition is reflected in the divergence of the susceptibility and this is independent of the nature of the inhomogeneous SC state, as it is the same for FF, PDW and bidirectional states, we need to investigate which one of these states actually has lower free energy. Since the phase transition is continuous we can investigate the stability of the different phase by expanding the free energy in powers of $\Delta_{\mathbf{Q}}$ up to fourth order.

B. Ginzburg Landau Free Energy

Considering all the possible SC aforementioned phases (FF and unidirectional and bidirectional PDW) the most general expression for the free energy compatible with gauge invariance, translation invariance and rotation in-

variance (or point group symmetry) has the form

$$\begin{aligned} F &= \frac{c_2}{2} (|\Delta_{\mathbf{Q}}|^2 + |\Delta_{-\mathbf{Q}}|^2 + |\Delta_{\bar{\mathbf{Q}}}|^2 + |\Delta_{-\bar{\mathbf{Q}}}|^2) \\ &\quad + \frac{c_4}{4} (|\Delta_{\mathbf{Q}}|^4 + |\Delta_{-\mathbf{Q}}|^4 + |\Delta_{\bar{\mathbf{Q}}}|^4 + |\Delta_{-\bar{\mathbf{Q}}}|^4) \\ &\quad + \frac{u}{4} (|\Delta_{\mathbf{Q}}|^2 |\Delta_{-\mathbf{Q}}|^2 + |\Delta_{\bar{\mathbf{Q}}}|^2 |\Delta_{-\bar{\mathbf{Q}}}|^2) \\ &\quad + \frac{v_1}{4} (|\Delta_{\mathbf{Q}}|^2 |\Delta_{\bar{\mathbf{Q}}}|^2 + |\Delta_{-\mathbf{Q}}|^2 |\Delta_{-\bar{\mathbf{Q}}}|^2) \\ &\quad + \frac{v_2}{4} (\Delta_{\mathbf{Q}} \Delta_{\bar{\mathbf{Q}}}^* \Delta_{-\mathbf{Q}} \Delta_{-\bar{\mathbf{Q}}}^* + \text{h.c.}) + \dots \end{aligned} \quad (4.17)$$

Knowing the expression for the coefficients in GL free energy we can see which state is favorable. This is equivalent to computing the GL free energy for each SC state and compare them to see which one is the lowest. For the FF state we will use that ansatz that the only non zero order parameter is $|\Delta_{\mathbf{Q}}|$, for the unidirectional PDW we will assume $|\Delta_{\mathbf{Q}}| = |\Delta_{-\mathbf{Q}}|$ and for the bidirectional (checkerboard) PDW $|\Delta_{\mathbf{Q}}| = |\Delta_{-\mathbf{Q}}| = |\Delta_{\bar{\mathbf{Q}}}| = |\Delta_{-\bar{\mathbf{Q}}}|$. Then for each state we have the following SC free energies

$$F_{\text{FF}} = \frac{c_2^{\text{FF}}}{2} |\Delta_{\mathbf{Q}}|^2 + \frac{c_4^{\text{FF}}}{4} |\Delta_{\mathbf{Q}}|^4 + \dots \quad (4.18)$$

$$F_{\text{PDW}} = \frac{c_2^{\text{PDW}}}{2} |\Delta_{\mathbf{Q}}|^2 + \frac{c_4^{\text{PDW}}}{4} |\Delta_{\mathbf{Q}}|^4 + \dots \quad (4.19)$$

$$F_{\text{Bi}} = \frac{c_2^{\text{Bi}}}{2} |\Delta_{\mathbf{Q}}|^2 + \frac{c_4^{\text{Bi}}}{4} |\Delta_{\mathbf{Q}}|^4 + \dots \quad (4.20)$$

where

$$\begin{aligned}
c_2^{\text{PDW}} &= 2c_2^{\text{FF}} \\
c_4^{\text{PDW}} &= 2c_4^{\text{FF}} + u \\
c_2^{\text{Bi}} &= 4c_2^{\text{FF}} \\
c_4^{\text{Bi}} &= 4c_4^{\text{FF}} + 2u + 2v_1 + 2v_2
\end{aligned} \quad (4.21)$$

where the coefficients are given in Appendix A. This expansion is only valid provided $c_4 > 0$. If $c_4 < 0$ we need to include higher order terms in the expansion to assure thermodynamic stability for large Δ_Q .

Using standard perturbation theory (see, e.g. Refs. [80–84]) the computation of the coefficients in the free energy reduces to a computation of a set of Feynman diagrams. An explicit derivation and form of the coefficients c_2 and c_4 for each of the SC states is given in Appendix A.

We find that for the range of parameters that we considered $c_4 > 0$. For $c_2 > 0$ the minimum is at $|\Delta_Q| = 0$, with $F = 0$. For $c_2 < 0$ the minimum is at $|\Delta_Q| = \sqrt{|c_2|/c_4}$, with $F = -c_2^2/4c_4$. Computing numerically the coefficients for the FF, and unidirectional and bidirectional PDW SC states we then compare their respective free energies resulting in the phase diagram shown in Fig. 4(a).

For $T/\Delta_{BCS} \gtrsim 0.33$, and provided the pairing coupling constant g_α^d is larger than its critical value, there is a continuous transition from the normal (Non-SC) state to the uniform $d_{x^2-y^2}$ -wave SC state, where the conventional BCS SC gap $\Delta_{BCS} = 2\omega_D \exp(-1/gN_F)$ is introduced to parametrize the dependence on g and ω_D , where ω_D is a high energy cutoff. For the isotropic state, $\delta = 0$, we recover the usual BCS second order transition at $T \approx 0.5669\Delta_{BCS}$. However, for $0.33 \gtrsim T/\Delta_{BCS} \gtrsim 0.23$ the transition from the normal (Non-SC) state to the uniform d -wave SC state is found to be first order, where there is a tricritical point, $T_{TCP} \simeq 0.33\Delta_{BCS}$.

The nodal directions of the $d_{x^2-y^2}$ -wave state are, as usual, along the diagonals. In the α phase these directions are symmetry directions where the two Fermi surfaces intersect each other, while the antinodal directions point along the lobes of the Fermi surface (see Fig. 4(b)). A putative d_{xy} -wave SC state would have its antinodal directions along the diagonals. However this state is not favored since the isotropic Fermi surface has been effectively gapped (except at a set of zero measure) leading, once again, to a state with a critical coupling constant. In addition, the d_{xy} form factor does not favor inhomogeneous SC states. We will not discuss this channel in what follows.

The most interesting part of the phase diagram is for $T/\Delta_{BCS} \lesssim 0.23$. In this region there is a continuous transition from the normal (Non-SC) state to an inhomogeneous superconducting state. Here we find two distinct phase transitions. For the temperature range $0.23 \gtrsim T/\Delta_{BCS} \gtrsim 0.20$ there is a continuous phase transition from the normal (Non-SC) state to a bidirectional

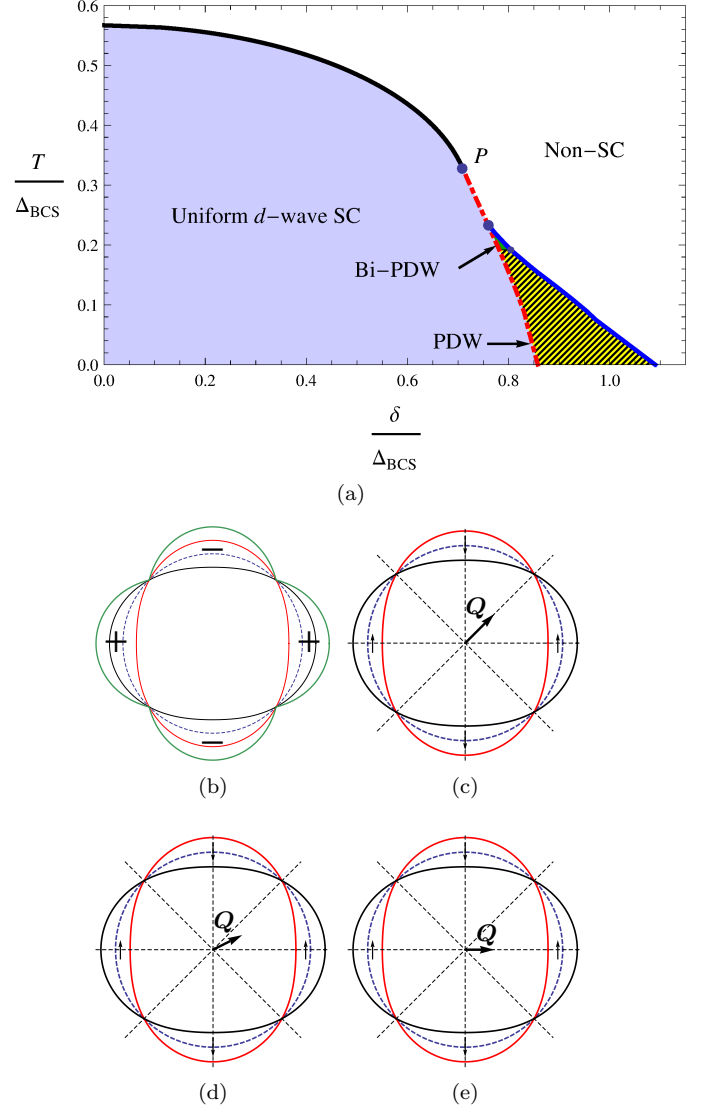


FIG. 4. (color online) (a) Phase diagram as a function of the parameters T/Δ_{BCS} and δ/Δ_{BCS} . The light blue region corresponds to the uniform d -wave SC state (with $Q = 0$). The yellow and green regions correspond to the PDW and the bidirectional SC states respectively. P labels the tricritical point discussed in the text. (b) Order parameter of the uniform d -wave SC state. (c)-(e) Direction of the inhomogeneous superconducting wave vector Q for (c) $0.23 \gtrsim T/\Delta_{BCS} \gtrsim 0.055$, (d) $0.055 \gtrsim T/\Delta_{BCS} > 0$ and (e) $T = 0$. Smooth curves show continuous phase transitions and first order transitions are shown as dashed curves.

PDW state, while for $T/\Delta_{BCS} \lesssim 0.20$ there is a continuous transition to a unidirectional PDW SC state. The ordering wave vector for the bidirectional PDW state is locked along the diagonal direction of the spin triplet nematic α phase (as shown in Fig. 4(c)).

In addition, we find a transition from the bi-directional PDW SC to the uniform d -wave SC state. Since our expansion for the free energy Eq. (4.17) is only valid close to the continuous transition, we investigate this transi-

tion using the exact expression for the free energy Eq. (4.16), and find that this transition is first order. However, the exact expression for the free energy Eq. (4.16) is valid only for the uniform and the FF SC states, and hence it can describe only the putative transition from the uniform d -wave to the FF state, depicted by a dashed curve in Fig. 4(a). The actual transition from the bidirectional PDW SC to the uniform d -wave SC state cannot be described by this free energy and it is most likely to occur to the left of the dashed curve.

We also investigated the possibility of coexistence of the inhomogeneous superconducting state and the uniform d -wave SC state. We found that this does not happen and that the system prefers to be either in the pure inhomogeneous superconducting state or in the pure uniform d -wave SC state. In addition our results suggest that the continuous phase transition from the normal α phase to the bidirectional PDW state merges with the first order transition into the d -wave state. This feature is not generic and it is likely to be an artifact of the model.

In our analysis we find that the direction (and magnitude) of the ordering wave vector \mathbf{Q} changes along the continuous phase boundary from the normal to the inhomogeneous superconducting state (as shown in Figs. 4(c), 4(d), and 4(e).) At $T = 0$, \mathbf{Q} points along the direction of maximum distortion $\phi = n\pi/2$, where $n \in \mathbb{Z}$ and $Q = |\mathbf{Q}| = 2\delta$ (Fig. 4(e)), and hence there are two possible orientations for the unidirectional PDW state. As the temperature increases, \mathbf{Q} rotates continuously towards the diagonal directions (Fig. 4(d)) and for $T/\Delta_{BCS} \gtrsim 0.055$ locks to the diagonal directions $\phi = n\pi/2 + \pi/4$ (Fig. 4(c)) where, at a somewhat higher temperature, the ordering becomes bidirectional along the two diagonals. In the intermediate regime there are four possible orientations for the unidirectional PDW state which reduce to two directions once the ordering wave vector locks along the diagonal direction of the α phase. We only find bidirectional PDW order along the principal axes of the α phase. A similar evolution of ordering wave vectors was found in studies of 2D FFLO phases due to the presence of a Zeeman magnetic field.⁵⁸

So far we have only considered an attractive pairing interaction in the d -wave channel. However, it is also possible to have spin-triplet superconductivity, e.g. p -wave, even if the microscopic interactions are nominally repulsive.^{85–87} In this case we can have pairing between fermions with the same spin polarization (up-up and down-down). As we can see from Fig. 1(a) there is perfect nesting, so there is an infinitesimal SC instability in the spin-triplet channel (with zero center of mass momentum of the Cooper pairs). This SC state is dominant for small values of the coupling constant. However, if the coupling constant in the d -wave channel is larger than a critical value g_c , it will be a competition between the d -wave SC state and the spin-triplet SC state. We considered possible coexistence and competition between both phases (d -wave and uniform p -wave). We found that

there is no coexistence between such phases and that the state with a larger T_c will be dominant. We can effectively tune the coupling constant in the d -wave channel in order make the d -wave SC state favorable against the p -wave SC state (or, equivalently, lower the coupling constant in the p -wave channel in order to decrease its T_c).

C. β -phase MF

We now turn to the case of the nematic triplet β phase and look for the possible superconducting states that may occur. For the choice $|\mathbf{n}_1| = |\mathbf{n}_2| = \bar{n}$ and $\mathbf{n}_1 = \bar{n}\hat{x}$ and $\mathbf{n}_2 = \bar{n}\hat{y}$ the Hamiltonian in the β phase can be written as:

$$H = \sum_{\mathbf{k}, \alpha, \beta} c_{\mathbf{k}, \alpha}^\dagger (\epsilon_{\mathbf{k}} - \bar{n} \mathbf{d}_{\mathbf{k}} \cdot \boldsymbol{\sigma}_{\alpha, \beta}) c_{\mathbf{k}, \beta} + \sum_{\mathbf{k}, \mathbf{k}', \mathbf{q}} V(\mathbf{k}, \mathbf{k}') c_{\mathbf{k}+\mathbf{q}/2, \uparrow}^\dagger c_{-\mathbf{k}+\mathbf{q}/2, \downarrow}^\dagger c_{-\mathbf{k}'+\mathbf{q}/2, \downarrow} c_{\mathbf{k}'+\mathbf{q}/2, \uparrow} \quad (4.22)$$

where $\mathbf{d}_{\mathbf{k}} = (\cos(l\theta_k), \sin(l\theta_k), 0)$.

$$\mathbf{d}_{\mathbf{k}} \cdot \boldsymbol{\sigma} = \begin{pmatrix} 0 & e^{-il\theta_k} \\ e^{il\theta_k} & 0 \end{pmatrix} \quad (4.23)$$

As we saw in the Section III, for the d -wave channel there are four equivalent directions for which the SC susceptibility for the β -phase has a maximum. Thus, as for the α -phase, we can focus on different cases for the superconducting order parameters: FF, PDW and bidirectional PDW. However, for s -wave pairing all the directions are equivalent, allowing us to have in principle orderings in all possible directions. Nevertheless, we will only study in addition to the three different states aforementioned, the tridirectional PDW and the triple helix state, which are expected to be favored on the basis of symmetry. These phases are defined as the follows

- Triple helix phase: in this phase three wave vectors contribute to the SC order parameter,

$$\Delta(\mathbf{r}) = \Delta_{\mathbf{Q}_1} e^{i\mathbf{Q}_1 \cdot \mathbf{r}} + \Delta_{\mathbf{Q}_2} e^{i\mathbf{Q}_2 \cdot \mathbf{r}} + \Delta_{\mathbf{Q}_3} e^{i\mathbf{Q}_3 \cdot \mathbf{r}} \quad (4.24)$$

where the angle between the \mathbf{Q}_i 's is $2\pi/3$. Assuming from now on $|\mathbf{Q}_1| = |\mathbf{Q}_2| = |\mathbf{Q}_3|$, so that $\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 = 0$, $|\Delta_{\mathbf{Q}_1}| = |\Delta_{\mathbf{Q}_2}| = |\Delta_{\mathbf{Q}_3}|$, and neglecting the phase fluctuations of these three complex order parameters, we write the previous expression as:

$$\Delta(\mathbf{r}) = \Delta_{|\mathbf{Q}_1|} (e^{i\mathbf{Q}_1 \cdot \mathbf{r}} + e^{i\mathbf{Q}_2 \cdot \mathbf{r}} + e^{i\mathbf{Q}_3 \cdot \mathbf{r}}) \quad (4.25)$$

- Tridirectional PDW phase: in this phase six wave vectors contribute to the SC order parameter,

$$\begin{aligned} \Delta(\mathbf{r}) = & \Delta_{\mathbf{Q}_1} e^{i\mathbf{Q}_1 \cdot \mathbf{r}} + \Delta_{-\mathbf{Q}_1} e^{-i\mathbf{Q}_1 \cdot \mathbf{r}} \\ & + \Delta_{\mathbf{Q}_2} e^{i\mathbf{Q}_2 \cdot \mathbf{r}} + \Delta_{-\mathbf{Q}_2} e^{-i\mathbf{Q}_2 \cdot \mathbf{r}} \\ & + \Delta_{\mathbf{Q}_3} e^{i\mathbf{Q}_3 \cdot \mathbf{r}} + \Delta_{-\mathbf{Q}_3} e^{-i\mathbf{Q}_3 \cdot \mathbf{r}} \end{aligned} \quad (4.26)$$

In this phase the SC order parameter is then a six-component complex field with $\Delta_{\pm\mathbf{Q}_i}$, where $i = 1, 2, 3$, being the six complex components (and hence six amplitudes and six phase fields). Under the assumption of parity and C_6 symmetry it reduces to

$$\Delta(\mathbf{r}) = 2|\Delta_{\mathbf{Q}}|(\cos(\mathbf{Q}_1 \cdot \mathbf{r}) + \cos(\mathbf{Q}_2 \cdot \mathbf{r}) + \cos(\mathbf{Q}_3 \cdot \mathbf{r})) \quad (4.27)$$

where we assumed that the tree ordering wave vector have the same magnitude, $|\mathbf{Q}_1| = |\mathbf{Q}_2| = |\mathbf{Q}_3| = |\mathbf{Q}|$ and that the angle between these vectors is $2\pi/3$. In addition we also assumed that $|\Delta_{\mathbf{Q}_1}| = |\Delta_{-\mathbf{Q}_1}| = |\Delta_{\mathbf{Q}_2}| = |\Delta_{-\mathbf{Q}_2}| = |\Delta_{\mathbf{Q}_3}| = |\Delta_{-\mathbf{Q}_3}|$

Since the possible SC phases for the β is larger than what we found in the case of the α phase, the associated Landau free energy has a more complex form. We will not exhibit it here in its full form. We will compute the free energy for each one of these phases in order to determine the phase diagram as we did for the α -phase. As for the α -phase we can perform a Hubbard-Stratonovich transformation to decouple the pairing interactions. Here too the calculation simplifies for the FF phases since the

Green function matrix is block diagonal, with each block being labeled by the momentum \mathbf{k} and the Matsubara frequency ω_n . Also, as what we found in the α phase, the free energy in the PDW SC states cannot be computed in closed form and can be obtained as a power series expansion in the PDW order parameters, whose coefficients need to be evaluated numerically. This analysis leads to the phase diagrams shown in Fig.5 and Fig. 6.

The action for the FF state is

$$S[\bar{\Psi}, \Psi, \Delta_{\mathbf{Q}}, \Delta_{\mathbf{Q}}^*] = - \sum_{\mathbf{k}, n} \bar{\Psi}_{\mathbf{k}, n} \mathcal{G}_{\mathbf{k}, i\omega_n}^{-1} \Psi_{\mathbf{k}, n} + \beta \frac{|\Delta_{\mathbf{Q}}|^2}{g} + \text{const.}, \quad (4.28)$$

where now the Nambu operator $\bar{\Psi}_{\mathbf{k}}$ is given by

$$\bar{\Psi}_{\mathbf{k}} = (\bar{\psi}_{\mathbf{k}+\mathbf{Q}/2, \uparrow}, \bar{\psi}_{\mathbf{k}+\mathbf{Q}/2, \downarrow}, \bar{\psi}_{-\mathbf{k}+\mathbf{Q}/2, \uparrow}, \bar{\psi}_{-\mathbf{k}+\mathbf{Q}/2, \downarrow}) \quad (4.29)$$

The inverse of the Green function for fixed \mathbf{k} and ω_n , $\mathcal{G}_{\mathbf{k}, i\omega_n}$, is given by the 4×4 matrix

$$\mathcal{G}_{\mathbf{k}, i\omega_n}^{-1} = \frac{1}{2} \begin{pmatrix} i\omega_n - \xi_{\mathbf{k}+\mathbf{Q}/2, \uparrow} & ne^{-il\theta_{\mathbf{k}+\mathbf{Q}/2}} & 0 & \Delta_{\mathbf{Q}}\gamma(\mathbf{k}) \\ ne^{il\theta_{\mathbf{k}+\mathbf{Q}/2}} & i\omega_n - \xi_{\mathbf{k}+\mathbf{Q}/2, \downarrow} & -\Delta_{\mathbf{Q}}^*\gamma(\mathbf{k}) & 0 \\ 0 & -\Delta_{\mathbf{Q}}\gamma(\mathbf{k}) & i\omega_n + \xi_{-\mathbf{k}+\mathbf{Q}/2, \uparrow} & -ne^{il\theta_{-\mathbf{k}+\mathbf{Q}/2}} \\ \Delta_{\mathbf{Q}}^*\gamma(\mathbf{k}) & 0 & -ne^{-il\theta_{-\mathbf{k}+\mathbf{Q}/2}} & i\omega_n + \xi_{-\mathbf{k}+\mathbf{Q}/2, \downarrow} \end{pmatrix} \quad (4.30)$$

In the β -phase $\xi_{\mathbf{k}, \uparrow} = \xi_{\mathbf{k}, \downarrow} = \xi_{\mathbf{k}}$ and $\xi_{\mathbf{k}} = \xi_{-\mathbf{k}}$. Also notice that $\theta_{-\mathbf{k}} = \theta_{\mathbf{k}} + \pi$.

After integrating out the fermionic degrees of freedom we get:

$$\begin{aligned} S_{\text{eff}}[\Delta_{\mathbf{Q}}^*, \Delta_{\mathbf{Q}}] &= - \sum_{\mathbf{k}, n} \ln \det \mathcal{G}_{\mathbf{k}, n}^{-1} + \beta \frac{|\Delta_{\mathbf{Q}}|^2}{g} + \text{const.} \\ &= - \sum_{\mathbf{k}, n} \sum_{j=1}^4 \ln \lambda_{\mathbf{k}, n}^{(j)} + \beta \frac{|\Delta_{\mathbf{Q}}|^2}{g} + \text{const.} \end{aligned} \quad (4.31)$$

where $\lambda_{\mathbf{k}, n}^{(j)}$ are the eigenvalues of $\mathcal{G}_{\mathbf{k}, i\omega_n}^{-1}$. Using that

$$\xi_{\pm\mathbf{k}+\mathbf{Q}/2} = \xi \pm Q/2 \cos(\theta - \phi) \quad (4.32)$$

we find that:

$$\lambda_{\mathbf{k}, n}^{(j)} = E_j - i\omega_n \quad (4.33)$$

where E_j are the eigenvalues of the 4×4 matrix $2\mathcal{G}_{\mathbf{k}, i\omega_n}^{-1} - i\omega_n I$.

Since we are interested in the inhomogeneous superconducting state, we focus from now on β phases with

even l . The Free energy in the β -phase is given by

$$\begin{aligned} F_s - F_n &= \frac{|\Delta_{\mathbf{Q}}|^2}{g} \\ &- TN(E_F) \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{\omega_D} d\xi \sum_{j=1}^4 \ln \left(\frac{1 + e^{-E_j(\Delta_{\mathbf{Q}})/T}}{1 + e^{-E_j(\Delta=0)/T}} \right) \end{aligned} \quad (4.34)$$

Once again, we now minimize the free energy with respect to Δ and \mathbf{Q} to find the equilibrium state. In contrast with the α phase, in the β -phase is not necessary to have d -wave pairing to have an inhomogeneous superconducting state. Thus, we can now have s - or d -wave pairing. We will study below the phase diagram for the β -phase for both SC channels.

The resulting phase diagrams have a rich structure. We find again that the transition from the normal (non-SC state) to the inhomogeneous superconducting state is continuous. However, in contrast to the α -phase case, we found that transition is continuous at all temperatures even as the inhomogeneous states meet the uniform states (s or d wave depending on the case). Thus in the β phase the transition from the normal (Non-SC) state to the uniform SC state and to the unidirectional PDW SC

state is continuous and are shown in the phase diagrams of Fig. 5 and Fig. 6 respectively.

In particular the transitions between the uniform SC states and the unidirectional PDW states are multicritical points which have the same structure as the well known Lifshitz points of magnetism and liquid crystals.⁸⁸ Near the Lifshitz points the SC susceptibilities can be expanded in powers of the magnitude of the ordering wave vector $Q = |\mathbf{Q}|$ in the form

$$\frac{\chi_\beta^\lambda(\mathbf{Q})}{N(E_F)} = \chi_0 + \chi_2 Q^2 + \chi_4 Q^4 + O(Q^6) \quad (4.35)$$

where the coefficients χ_0 , χ_2 and χ_4 need to be computed numerically. The important feature of the SC susceptibility is that the coefficient χ_2 continuously changes along the phase boundary between the normal state and the uniform SC state from positive to negative values across the Lifshitz point where it vanishes. The other two coefficients, χ_0 and χ_4 , also vary smoothly but without changing sign.

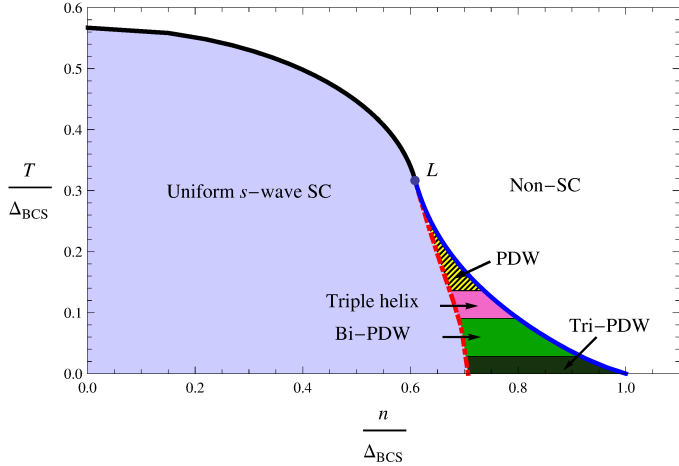


FIG. 5. (color online) Phase diagram in the nematic β -phase for s -wave pairing. Here n is the magnitude of the order parameter of the spin triplet nematic β phase. All transitions from the non-SC β phase are continuous. Notice the complex sequence of phases with different types of homogeneous and inhomogeneous orders, and their sequence of multicritical points. Here L labels the Lifshitz point discussed in the text. Smooth curves show continuous phase transitions and first order transitions are shown as dashed curves. The same caveats we pointed out in the case of the α phase apply here too. See text for details.

As we did for the α -phase we can compute the Ginzburg-Landau free energy for the β -phase. We need again to determine the coefficients c_2 and c_4 in the GL free energy for the different SC phases. This can be done in the same way we determined the GL free energy coefficients for the α -phase (explained in detail in the appendix A). In addition, as it was mentioned above, there is a parallel in the calculations for the expression with even l in

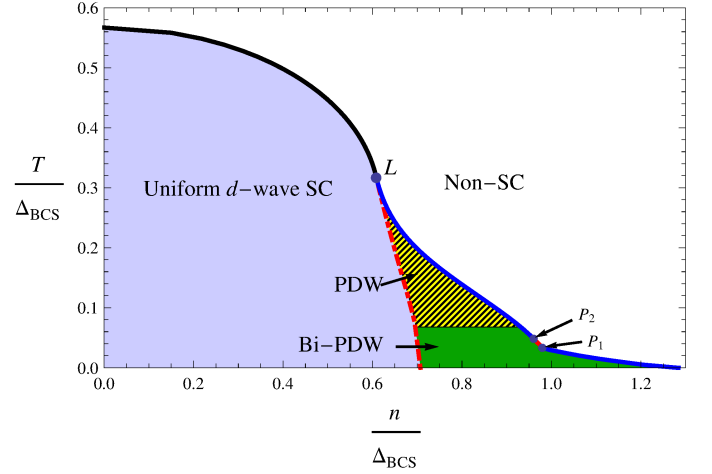


FIG. 6. (color online) Phase diagram in the nematic β -phase for d -wave pairing. Here n is the magnitude of the order parameter of the spin-triplet nematic β phase. L is the Lifshitz point separating the uniform d -wave SC from the unidirectional PDW phase. P_2 is a tricritical point separating a continuous non-SC high temperature phase from the bidirectional PDW phase, and P_1 is a bicritical point separating the low temperature bidirectional phase oriented along the anti-nodal direction of the d -wave from a bidirectional PDW oriented along the nodal direction. Smooth curves show continuous phase transitions and first order transitions are shown as dashed curves. See text for details.

the β -phase and the calculations carried by Shimahara in Ref. 56. Therefore we can use his results to determine the phase diagrams for the s - and d -wave channels.

For the s -wave pairing we found the following. For $0.318 \gtrsim T/\Delta_{BCS} \gtrsim 0.136$ the favored SC state is a unidirectional PDW state with wave vector \mathbf{Q} with arbitrary direction. The unidirectional PDW SC phase meets the uniform s -wave SC at the Lifshitz point shown in Fig.5. The wave vector of the unidirectional PDW state grows continuously from zero away from the Lifshitz point.⁸⁸ In the temperature range $0.136 \gtrsim T/\Delta_{BCS} \gtrsim 0.091$ a triple helix FF state is favorable, while for $0.091 \gtrsim T/\Delta_{BCS} \gtrsim 0.028$ the bidirectional state is favorable. Finally, for $T/\Delta_{BCS} \lesssim 0.028$ a tridirectional PDW state is the favored. These results are summarized in the phase diagram in Fig. 5.

In the case of d -wave pairing the situation is different. The phase diagram for the β phase in the d -wave case is shown in Fig. 6. Here we find that the direction of ordering wave vector \mathbf{Q} of the inhomogeneous SC states is no longer arbitrary and it is not same throughout these superconducting states. For $T/\Delta_{BCS} \lesssim 0.034$, the ordering wave vector \mathbf{Q} points in the antinodal directions of the d -wave with $\phi = n\pi/2$, where $n \in \mathbb{Z}$. On the other hand, for $T/\Delta_{BCS} \gtrsim 0.034$, the ordering wave vector \mathbf{Q} now points in the nodal directions with $\phi = n\pi/2 + \pi/4$.

For $T/\Delta_{BCS} \lesssim 0.034$ we find a bidirectional PDW state whose ordering wave vector \mathbf{Q} points along the

antinodal directions of the d -wave. For $0.034 \lesssim T/\Delta_{BCS} < 0.068$ we find a bidirectional PDW state whose ordering wave vector \mathbf{Q} points along the nodal directions of the d -wave. This means that above the tricritical point P_1 shown in Fig.6 (where $T/\Delta_{BCS} = 0.034$) the direction of the ordering wave vector \mathbf{Q} rotates by $\pi/4$ and above P_1 points along the nodal directions of the d -wave. Above the tricritical point P_1 the transition from the normal state (Non-SC) to the bidirectional PDW state is first order and becomes continuous at the second tricritical point P_2 at $T/\Delta_{BCS} = 0.048$. The coefficient c_4 of the Landau free energy vanishes at both tricritical points (as it should). For $0.068 \lesssim T/\Delta_{BCS} \lesssim 0.318$ we find a unidirectional PDW state whose ordering wave vector \mathbf{Q} points along the nodal directions of the d -wave.

V. CONCLUDING REMARKS

A principal motivation for this work was to investigate using controlled approximations the possible relation between electronic liquid crystal phases (of which the spin triplet nematic states are just two examples) and superconductivity. Our results indicate that this type of electronic liquid crystal phases naturally give rise to complex inhomogeneous superconducting phases. Unfortunately, so far as we know, spin-triplet nematic metallic phases have yet to be discovered in experiment.

Earlier studies of superconducting instabilities in spin-singlet nematic phases did not reach a clear answer.^{62,89} In addition, PDW phases and others of similar nature, are notoriously difficult to study as they are outside the reach of weak coupling BCS theory.^{42,90} On the other hand, high-quality numerical tensor-network approaches such as iPEPS have provided solid evidence for the existence (or, at least, competitiveness) of PDW phases in simple 2D strongly correlated systems such as the $t - J$ model.⁴⁹ Interestingly these authors find that some degree of fixed nematicity (i.e. explicit rotational symmetry breaking) strongly favors PDW ordered SC states. In addition, commensurate PDW phases have been shown to occur in Kondo-Heisenberg chains⁵² and in doped spin-ladders.⁵³

The models we studied in this paper are, on the other hand, too idealized as they stand to be relevant to the physics of the cuprates. In addition of the important role of magnetism that these spin-triplet phases imply, for which there is no evidence in these materials, we have used a continuum description with simple (nearly circular) Fermi surfaces. In strongly correlated systems rotational spatial symmetry is strongly broken down to the point group symmetry of the lattice. So far there are few studies of lattice models with nematic spin-triplet phases^{68,69} and they typically find that this Pomeranchuk type phase transition requires a substantial value of the one-site Hubbard interactions which put them outside the regime in which their mean field theories may be reliable. Nevertheless these results are interesting and

suggest that PDW type phases may also arise in these models.

Using weak coupling BCS-type methods we showed that nematic spin triplet α and β phases give rise to a complex phase diagram which includes pair-density-wave phases and other spatially inhomogeneous superconducting states. Rather than considering a specific microscopic model we used instead effective pairing interactions in different channels (s , p and d wave), with effective coupling constants for each, and investigated what superconducting states arose as instabilities of the α and β spin-triplet nematic states. The theory is well controlled by tuning to the nematic spin triplet to normal Fermi liquid quantum phase transition. The distance to this quantum critical point inside the nematic spin triplet states plays the role of the small parameter which justifies the use of weak coupling mean field theory (BCS) to describe the resulting superconducting states. In this sense, the nematic spin triplet quantum phase transition can be regarded as a complex multicritical point.

An important feature of the phase diagrams that we present here is that the critical temperatures of all the phases have comparable magnitude. This is the consequence of having fine-tuned to the spin-triplet nematic quantum critical point. A puzzling feature of the intertwined orders seen in the experiments in the cuprate superconductors is that the critical temperatures have the same typical magnitude over a substantial range of doping and for rather different materials. It is unreasonable to think that all the cuprate superconductors have conspired to be fine-tuned to a multicritical point as in the calculation that we have done here. Rather this is presumably a consequence of strong correlation physics as in the recent work of Corboz, Rice and Troyer⁴⁹.

Finally, in this paper we assumed that we were deep enough in the spin-triplet nematic phase that its quantum fluctuations can be neglected. This is clearly not the case close enough to the quantum phase transition from the Fermi liquid phase. In addition we also neglected the possible role of Goldstone modes of the nematic triplet state. These modes, which are gapped by lattice effects, may drive the fermionic fluid into a non-Fermi liquid regime in their absence and change the physics of the superconducting state in an essential way. In particular, in the absence of lattice effects, the Goldstone modes may invalidate the use of BCS-type schemes which require the existence of sharply defined quasiparticles.

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Appendix A: Coefficients in the Ginzburg Landau Free Energy

In the following we give a detailed derivation for the coefficients in the GL free energy (4.17) for the α -phase. Similar analysis can be carried out for the β -phase. Following Radzihovsky⁸⁴ (and references therein), we write

$$S = S_0 + S_{\text{int}} + \beta \sum_{\mathbf{Q}} \frac{|\Delta_{\mathbf{Q}}|^2}{g}, \text{ where:}$$

$$\begin{aligned} S_0 &= \int_0^\beta d\tau \sum_{\mathbf{k}, \sigma} \bar{\psi}_{\mathbf{k}, \sigma} (\partial_\tau + \xi_{\mathbf{k}, \sigma}) \psi_{\mathbf{k}, \sigma} \\ S_{\text{int}} &= - \int_0^\beta d\tau \sum_{\mathbf{Q}} \sum_{\mathbf{k}} [\gamma(\mathbf{k}) \bar{\psi}_{\mathbf{k}+\mathbf{Q}/2, \uparrow} \bar{\psi}_{-\mathbf{k}+\mathbf{Q}/2, \downarrow} \Delta_{\mathbf{Q}} + \gamma(\mathbf{k}) \Delta_{\mathbf{Q}}^* \psi_{-\mathbf{k}+\mathbf{Q}/2, \downarrow} \psi_{\mathbf{k}+\mathbf{Q}/2, \uparrow}] \end{aligned} \quad (\text{A1})$$

The effective action is then given by:

$$e^{-S_{\text{eff}}} = \int D\psi D\bar{\psi} e^{-S} = e^{-\beta \sum_{\mathbf{Q}} \frac{|\Delta_{\mathbf{Q}}|^2}{g}} \int D\psi D\bar{\psi} e^{-S_0} e^{-S_{\text{int}}} \quad (\text{A2})$$

We can then expand in powers of S_{int} we obtain

$$\begin{aligned} \int D\psi D\bar{\psi} e^{-S_0} e^{-S_{\text{int}}} &= Z_0 \times \left[1 + \frac{1}{2!} \langle S_{\text{int}}^2 \rangle_0 + \frac{1}{4!} \langle S_{\text{int}}^4 \rangle_0 + \dots \right] \\ &= Z_0 \times \exp \left(\frac{1}{2!} \langle S_{\text{int}}^2 \rangle_0^c + \frac{1}{4!} \langle S_{\text{int}}^4 \rangle_0^c + \dots \right) \end{aligned} \quad (\text{A3})$$

where we denoted by $\langle A \rangle_0^c$ the connected expectation value in the normal state and where we used the notation

$$Z_0 \equiv \int D\psi D\bar{\psi} e^{-S_0}, \quad \langle \dots \rangle_0 \equiv \frac{1}{Z_0} \int D\psi D\bar{\psi} e^{-S_0} (\dots) \quad (\text{A4})$$

We also used the fact that the expectation value of odd powers of the interacting part of the action vanishes in the normal state, $\langle S_{\text{int}}^{2p+1} \rangle_0 = 0$.

We will now focus in the quadratic and quartic terms for the FF, the LO (PDW) and the bidirectional states:

1. FF state.

i) Quadratic term c_2^{FF} .

For the quadratic term we need to compute $\frac{1}{2!} \langle S_{\text{int}}^2 \rangle_0$, which can be represented by the Feynman diagram shown in Fig. 7. This diagram corresponds to the superconducting susceptibility. Adding the term $\frac{|\Delta_{\mathbf{Q}}|^2}{g}$

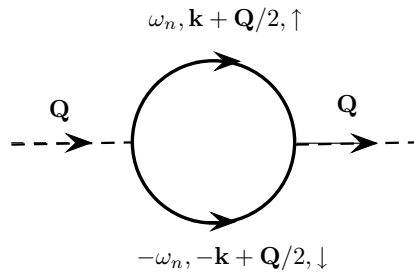


FIG. 7. Superconducting susceptibility for the FF state.

to the effective action and using that $\frac{1}{gN(E_F)} = \ln\left(\frac{2\omega_D}{\Delta_{BCS}}\right)$ we can write c_2^{FF} as:

$$\frac{c_2^{\text{FF}}}{N(E_F)} = -2\ln\left(\frac{1}{4\pi T}\right) + 2\text{Re} \int_0^{2\pi} \frac{d\theta}{2\pi} 2\cos^2(2\theta)\psi\left(\frac{1}{2} + i\frac{\delta}{2\pi T}\cos(2\theta) - i\frac{Q/2}{2\pi T}\cos(\theta - \phi)\right) \quad (\text{A5})$$

where T , Q and δ are in units of Δ_{BCS} , and $\psi(z)$ is the digamma function (for $z \in \mathbb{C}$)

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) \quad (\text{A6})$$

where

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t} \quad (\text{A7})$$

is the Euler Gamma function. Below we use the standard notation $\psi^{(n)}(z)$ for the derivatives of the digamma function.

ii) Quartic term c_4^{FF} .

For the quartic term we need to compute

$$\frac{1}{4!} \langle S_{\text{int}}^4 \rangle_0^c = \frac{1}{4!} (\langle S_{\text{int}}^4 \rangle_0 - 3\langle S_{\text{int}}^2 \rangle_0^2) \quad (\text{A8})$$

which can be represented by the diagram of Fig. 8. The algebraic expression for this diagram is given by:

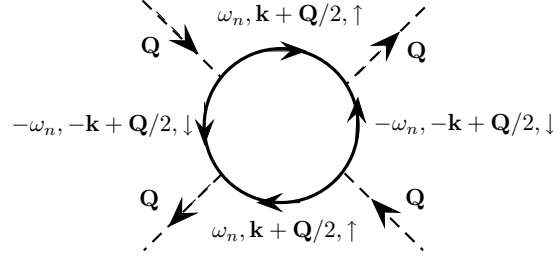


FIG. 8. Quartic effective interaction for the FF state.

$$\frac{c_4^{\text{FF}}}{N(E_F)} = -\frac{1}{8\pi^2 T^2} \int_0^{2\pi} \frac{d\theta}{2\pi} 4\cos^4(2\theta) \text{Re} \left[\psi^{(2)} \left(\frac{1}{2} + i\frac{\delta}{2\pi T}\cos 2\theta - i\frac{Q/2}{2\pi T}\cos(\theta - \phi) \right) \right] \quad (\text{A9})$$

2. Unidirectional PDW state.

i) Quadratic term c_2^{PDW} .

For the quadratic term we need to compute $\frac{1}{2!} \langle S_{\text{int}}^2 \rangle_0$, which can be represented by the diagrams of Fig. 9. Each diagram produce the same contribution. Now remember than in the PDW state we have two plane waves with wave vectors \mathbf{Q} and $-\mathbf{Q}$, so $\sum_{\mathbf{q}} \frac{|\Delta_{\mathbf{q}}|^2}{g} = 2 \frac{|\Delta_{\mathbf{Q}}|^2}{g}$ where we used that $|\Delta_{\mathbf{Q}}| = |\Delta_{-\mathbf{Q}}|$, so we have that

$$\frac{c_2^{\text{PDW}}}{N(E_F)} = 2 \frac{c_2^{\text{FF}}}{N(E_F)} \quad (\text{A10})$$

where again T , Q and δ are in units of Δ_{BCS}

ii) Quartic term c_4^{PDW} .

For the quadratic term we need the diagrams in Fig. 10. The algebraic expression for the diagram in Fig. 10a is given by Eq. (A9). We need only to compute the diagram in Fig. 10b. The expression for this diagram is given by:

$$I_2 = N(E_F) \int_0^{2\pi} d\theta \frac{\cos^4 2\theta}{8\pi^2 T Q/2 \cos(\theta - \phi)} \left\{ \text{Im} \left[\psi^{(1)} \left(\frac{1}{2} - i\frac{\delta}{2\pi T}\cos 2\theta - i\frac{Q/2}{2\pi T}\cos(\theta - \phi) \right) \right] - \text{Im} \left[\psi^{(1)} \left(\frac{1}{2} - i\frac{\delta}{2\pi T}\cos 2\theta + i\frac{Q/2}{2\pi T}\cos(\theta - \phi) \right) \right] \right\} \quad (\text{A11})$$

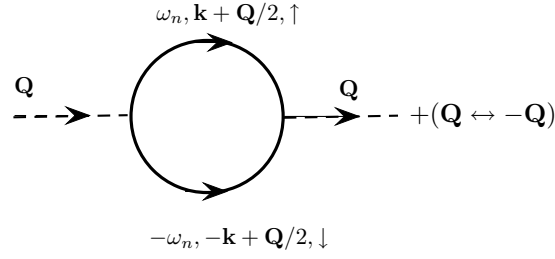
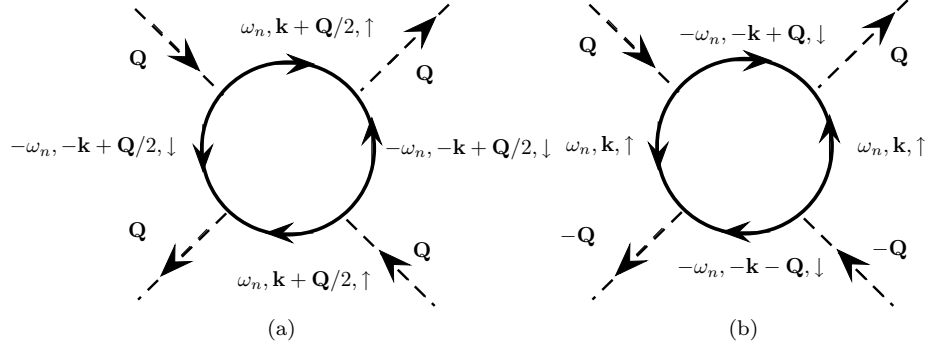


FIG. 9. Superconducting susceptibility for the LO state.

FIG. 10. Diagrams that contribute to the coefficient C_4^{PDW} .

We can write then:

$$\frac{c_4^{\text{PDW}}}{N(E_F)} = 2 \frac{c_4^{\text{FF}}}{N(E_F)} + 4I_2 \quad (\text{A12})$$

where the factor of two in the first term comes from $\mathbf{Q} \rightarrow -\mathbf{Q}$ in the diagram in Fig. 10(a) and the factor of 4 in the second term is computed in a similar way.

3. Bidirectional PDW state

i) Quadratic term c_2^{Bi} .

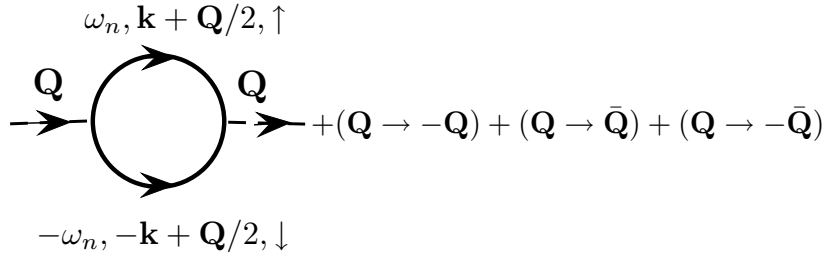


FIG. 11. Susceptibility for bidirectional superconducting order.

For the quadratic term we need to compute $\frac{1}{2!} \langle S_{\text{int}}^2 \rangle_0$, which can be represented by the diagram of Fig. 11, where $\bar{\mathbf{Q}} = R_{\pi/2} \mathbf{Q}$ is the wave vector \mathbf{Q} rotated by $\pi/2$. Each term yields the same contribution.

Now remember that the bidirectional state have four plane waves with wave vectors \mathbf{Q} , $-\mathbf{Q}$, $\bar{\mathbf{Q}}$, $-\bar{\mathbf{Q}}$ so $\sum_{\mathbf{q}} \frac{|\Delta_{\mathbf{q}}|^2}{g} = 4 \frac{|\Delta_{\mathbf{Q}}|^2}{g}$ where we used that, in order to minimize the free energy, $|\Delta_{\mathbf{Q}}| = |\Delta_{-\mathbf{Q}}| = |\Delta_{\bar{\mathbf{Q}}}| = |\Delta_{-\bar{\mathbf{Q}}}|$. Therefore we find

$$\frac{c_2^{\text{Bi}}}{N(E_F)} = 4 \frac{c_2^{\text{FF}}}{N(E_F)} \quad (\text{A13})$$

where T , Q and δ are in units of the BCS gap Δ_{BCS}

ii) Quartic term c_4^{Bi} .

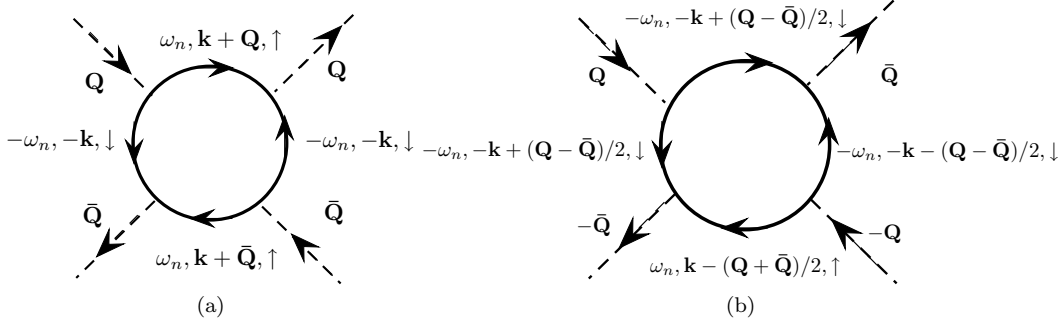


FIG. 12.

For the quartic term we need several diagrams. First we have contributions from the diagrams in Fig. 10(a) and Fig. 10(b), to which we must add the contributions from the ordering wave vector along the rotated direction, $\mathbf{Q} \rightarrow \bar{\mathbf{Q}}$.

We also have the two new diagrams shown in Fig. 12(a) and Fig. 12(b), whose analytic expressions are

$$\begin{aligned} \frac{c_{4B1}}{N(E_F)} &= \frac{1}{4\pi QT} \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{4 \cos^4(2\theta)}{(\cos(\theta - \phi) - \sin(\theta - \phi))} \left\{ \text{Im} \left[\psi^{(1)} \left(\frac{1}{2} - i \frac{\delta}{2\pi T} \cos 2\theta + i \frac{Q/2}{2\pi T} \sin(\theta - \phi) \right) \right] \right. \\ &\quad \left. - \text{Im} \left[\psi^{(1)} \left(\frac{1}{2} - i \frac{\delta}{2\pi T} \cos 2\theta + i \frac{Q/2}{2\pi T} \cos(\theta - \phi) \right) \right] \right\} \\ \frac{c_{4B2}}{N(E_F)} &= \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{4 \cos^4(2\theta)}{Q^2 (\sin^2(\theta - \phi) - \cos^2(\theta - \phi))} \text{Re} \left\{ \psi \left(\frac{1}{2} + i \frac{\delta}{2\pi T} \cos 2\theta + i \frac{Q/2}{2\pi T} \sin(\theta - \phi) \right) \right. \\ &\quad + \psi \left(\frac{1}{2} + i \frac{\delta}{2\pi T} \cos 2\theta - i \frac{Q/2}{2\pi T} \sin(\theta - \phi) \right) \\ &\quad - \psi \left(\frac{1}{2} + i \frac{\delta}{2\pi T} \cos 2\theta + i \frac{Q/2}{2\pi T} \cos(\theta - \phi) \right) \\ &\quad \left. - \psi \left(\frac{1}{2} + i \frac{\delta}{2\pi T} \cos 2\theta - i \frac{Q/2}{2\pi T} \cos(\theta - \phi) \right) \right\} \end{aligned} \quad (\text{A14})$$

We can then write

$$\frac{c_4^{\text{Bi}}}{N(E_F)} = \frac{4}{N(E_F)} (c_4^{\text{FF}} + 2I_2 + 4c_{4B1} + 2c_{4B2}) \quad (\text{A15})$$

Now that we have computed the coefficients for the FF, PDW and Bidirectional states we need to see which one has less energy. Using that $F_{\min} = -\frac{c_2^2}{4c_4}$, we have that:

$$F_{\min}^{\text{FF}} = -\frac{(c_2^{\text{FF}})^2}{4c_4^{\text{FF}}} = -\frac{(c_2^{\text{FF}})^2}{4} \frac{1}{c_4^{\text{FF}}} \quad (\text{A16})$$

$$F_{\min}^{\text{PDW}} = -\frac{(c_2^{\text{PDW}})^2}{4c_4^{\text{PDW}}} = -\frac{(2c_2^{\text{FF}})^2}{4(2c_4^{\text{FF}} + 4I_2)} = -\frac{(c_2^{\text{FF}})^2}{4} \frac{2}{(c_4^{\text{FF}} + 2I_2)} \quad (\text{A17})$$

$$F_{\min}^{\text{Bi}} = -\frac{(c_2^{\text{Bi}})^2}{4c_4^{\text{Bi}}} = -\frac{(4c_2^{\text{FF}})^2}{4 \cdot 4(c_4^{\text{FF}} + 2I_2 + 4c_{4B1} + 2c_{4B2})} = -\frac{(c_2^{\text{FF}})^2}{4} \frac{4}{(c_4^{\text{FF}} + 2I_2 + 4c_{4B1} + 2c_{4B2})} \quad (\text{A18})$$

In order to see which state has less energy we need to compute numerically $\frac{1}{c_4^{\text{FF}}}$, $\frac{2}{(c_4^{\text{FF}} + 2I_2)}$ and

$\frac{4}{(c_4^{\text{FF}} + 2I_2 + 4c_{4B1} + 2c_{4B2})}$ and see which of these terms is larger. As it was mentioned above, we found that the unidirectional and bidirectional PDW states have less energy than the FF state. In addition, we found that for $T \gtrsim 0.2$ the bidirectional state has less energy than the unidirectional PDW state.

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